EXAMPLE: Asthma Example: Let Y be the random variable recording the number of children (out of $n = 15$) that are observed to be sufferers.

$Y\|\theta \sim Binomial(n, \theta)$	$f_{Y\|\theta}(y;\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
$\theta \sim Beta(\alpha, \beta)$	$\pi_\theta(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$
$\theta\|Y = y \sim Beta(\alpha + y, \beta + n - y)$	$\pi_{\theta\|Y}(\theta\|y) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+y)\Gamma(\beta+n-y)}\theta^{y+\alpha-1}(1-\theta)^{\beta+n-y-1}$

Note that

$$\begin{aligned}
f_Y(y) &= \int f_{Y|\theta}(y;\theta)\pi_\theta(\theta)d\theta \\
&= \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{y+\alpha-1}(1-\theta)^{\beta+n-y-1}d\theta \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+y)\Gamma(\beta+n-y)}{\Gamma(\alpha+\beta+n)}
\end{aligned}$$

INTEGRATION PROBLEMS IN BAYESIAN INFERENCE

BAYES CALCULATION

$$\pi_{\theta|Y}(\theta|y) = \frac{f_{Y|\theta}(y;\theta)\pi(\theta)}{f_Y(y)} = \frac{f_{Y|\theta}(y;\theta)\pi(\theta)}{\int f_{Y|\theta}(y;\theta)\pi(\theta)d\theta}$$

POSTERIOR EXPECTATION

$$\mathrm{E}_{\theta|Y}\left[\theta|y\right] = \int \theta\pi_{\theta|Y}(\theta|y)d\theta$$

MARGINAL LIKELIHOOD

$$f_Y(y) = \int f_{Y|\theta}(y;\theta)\pi(\theta)d\theta$$

MARGINALIZATION

$$\pi_{\varphi|Y}(\varphi|y) = \int \pi_{\theta,\varphi|Y}(\theta,\varphi|y)d\theta$$

PREDICTION

$$f_{Y^*|Y}(y^*|y) = \int f_{Y^*|\theta}(y^*;\theta)\pi_{\theta|Y}(\theta|y)d\theta$$

all require **INTEGRATION**

$$I = E_{\theta|Y}\left[h(\theta)|y\right] = \int h(\theta)\pi_{\theta|Y}(\theta|y)d\theta$$

Approaches:

- ANALYTIC
- ANALYTIC APPROXIMATION
- NUMERICAL APPROXIMATION
- SIMULATION-BASED (MONTE CARLO)

MONTE CARLO

$$I = E_{\theta|Y}\left[h(\theta)|y\right] = \int h(\theta)\pi_{\theta|Y}(\theta|y)d\theta \approx \frac{1}{N}\sum_{i=1}^{N} h\left(\theta^{(i)}\right)$$

where $\theta^{(1)}, ..., \theta^{(N)}$ are samples from $\pi_{\theta|Y}(\theta|y)$**MONTE CARLO.**

$$\frac{1}{N}\sum_{i=1}^{N} h\left(\theta^{(i)}\right) \xrightarrow{P} I$$

Could use

- PSEUDORANDOM samples or
- QUASIRANDOM samples (QUASI-MONTE CARLO)

IMPORTANCE SAMPLING

$$I = E_{\theta|Y}\left[h(\theta)|y\right] = \int h(\theta)\pi_{\theta|Y}(\theta|y)d\theta = \int h(\theta)\left\{\frac{\pi_{\theta|Y}(\theta|y)}{g(\theta)}\right\}g(\theta)d\theta \approx \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{h\left(\theta^{(i)}\right)\pi_{\theta|Y}(\theta^{(i)}|y)}{g\left(\theta^{(i)}\right)}\right\}$$

where $\theta^{(1)}, ..., \theta^{(N)}$ are samples from $g(\theta)$.