M3S3/M4S3 : SOLUTIONS 2

1. To establish a.s. convergence, apart from considering the original definition directly, we might consider three possible methods of proof;

I the equivalent characterization

$$X_n \xrightarrow{a.s.} X \qquad \Longleftrightarrow \qquad \lim_{n \longrightarrow \infty} P\,[\,|X_m - X| < \epsilon,\ \forall\, m \geq n\,] = 1 \qquad \text{for each } \epsilon > 0.$$

II the Borel-Cantelli Lemma

III the consequence of "fast enough" convergence in probability or rth mean.

It transpires that we have insufficient information to prove whether or not each of the sequences converges almost surely to any specific limit. For example, in each case

$$\sum_{n=1}^{\infty} P[X_n = c] = \infty$$

for all c, which begins to imply a.s. convergence, but the crucial condition of independence is not necessarily met. Also, it is not possible usefully to bound $P\,[\,|X_m - X| < \epsilon,\ \forall\, m \geq n\,]$.

(a) Clearly if the sequence converges, it converges to 1 or 2, and as $n \longrightarrow \infty$ it is clear that the probability $P[X_n = 1] \longrightarrow 0$, so we check whether the limit is 2.

We have

$$E\left[|X_n - 2|^2\right] = \left(|-1|^2 \times \frac{1}{n}\right) + \left(|0|^2 \times \frac{n-1}{n}\right) = \frac{1}{n} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

so $X_n \xrightarrow{r=2} 2$; we can also prove directly that, for $\epsilon > 0$,

$$P[|X_n - 2| < \epsilon] = P[X_n = 2] = 1 - \frac{1}{n} \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty$$

so $X_n \xrightarrow{p} 2$ (although this does follow because of the convergence in $r = 2$ mean).

(b) Here it seems that X_n may converge to 1; we have

$$E\left[|X_n - 1|^2\right] = \left(|n^2 - 1|^2 \times \frac{1}{n}\right) + \left(|0|^2 \times \frac{n-1}{n}\right) = \frac{(n^2-1)^2}{n} \nrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

so X_n does not converge in $r = 2$ mean to 1; by similar arguments, it can be shown that X_n does not converge in this mode to any fixed constant. However, we can prove that, for $\epsilon > 0$,

$$P[|X_n - 1| < \epsilon] = P[X_n = 1] = 1 - \frac{1}{n} \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty \qquad \therefore X_n \xrightarrow{p} 1.$$

(c) Here it seems that X_n may converge to 0; we have

$$E\left[|X_n - 0|^2\right] = \left(|n|^2 \times \frac{1}{\log n}\right) + \left(|0|^2 \times 1 - \frac{1}{\log n}\right) = \frac{n^2}{\log n} \nrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

so X_n does not converge in $r = 2$ mean to 0; by similar arguments, it can be shown that X_n does not converge in this mode to any fixed constant. However, for $\epsilon > 0$,

$$P[|X_n - 0| < \epsilon] = P[X_n = 0] = 1 - \frac{1}{\log n} \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty \quad X_n \xrightarrow{p} 0.$$

2. By assumption

$$\lim_{n\longrightarrow\infty} E\left[\,|X_n - X|^2\,\right] = \lim_{n\longrightarrow\infty} E\left[\,|Y_n - Y|^2\,\right] = 0$$

Then, by the Cauchy-Schwarz (and hence the triangle) inequality,

$$|Z_n - Z|^2 = |X_n + Y_n - X - Y|^2 = |(X_n - X) + (Y_n - Y)|^2 \le |X_n - X|^2 + |Y_n - Y|^2$$

and taking expectations, and limits as $n \longrightarrow \infty$ yields the result, that is

$$E\left[|Z_n - Z|^2\right] \le E\left[|X_n - X|^2\right] + E\left[|Y_n - Y|^2\right] \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

For convergence in probability, fix $\epsilon > 0$; then, by assumption

$$\lim_{n\longrightarrow\infty} P[|X_n - X| < \epsilon/2] = 1 \qquad \lim_{n\longrightarrow\infty} P[|Y_n - Y| < \epsilon/2] = 1$$

so that

$$\lim_{n\longrightarrow\infty} P[\,|X_n - X| + |Y_n - Y| < \epsilon\,] = 1.$$

Now

$$|X_n + Y_n - X - Y| \le |X_n - X| + |Y_n - Y| \tag{1}$$

and hence

$$|X_n - X| + |Y_n - Y| < \epsilon \quad \Longrightarrow \quad |X_n + Y_n - X - Y| < \epsilon \tag{2}$$

therefore

$$P[\,|X_n - X| + |Y_n - Y| < \epsilon\,] \le P[\,|X_n + Y_n - X - Y| < \epsilon\,].$$

As $n \longrightarrow \infty$,

$$P[\,|X_n - X| + |Y_n - Y| < \epsilon\,] \longrightarrow 1 \quad \Longrightarrow \quad P[\,|X_n + Y_n - X - Y| < \epsilon\,] = P[\,|Z_n - Z|\,] \longrightarrow 1$$

and $Z_n \xrightarrow{p} Z$.

For convergence almost surely, fix $\epsilon > 0$; then, by assumption,

$$\lim_{n\longrightarrow\infty} P[|X_m - X| < \epsilon/2\,, \forall\, m \ge n] = \lim_{n\longrightarrow\infty} P[|Y_m - Y| < \epsilon/2\,, \forall\, m \ge n] = 1$$

Now, recall the definition of the limit L of a real sequence $\{a_n\}$; for every $\epsilon > 0$ there exists a natural number n_0 such that for all $n > n_0$, $|a_n - L| < \epsilon$. This implies here that we can find an n large enough such that

$$P[|X_m - X| < \epsilon/2\,, \forall\, m \ge n] \quad \text{and} \quad P[|Y_m - Y| < \epsilon/2\,, \forall\, m \ge n]$$

and hence

$$P[|X_m - X| < \epsilon/2 \text{ and } |Y_m - Y| < \epsilon/2,\ \forall\, m \ge n]$$

are arbitrarily close to 1. But

$$|X_m - X| < \epsilon/2 \text{ and } |Y_m - Y| < \epsilon/2 \quad \Longrightarrow \quad |X_m - X| + |Y_m - Y| < \epsilon$$

for all $m \ge n$. Therefore

$$P[|X_m - X| + |Y_m - Y| < \epsilon,\ \forall\, m \ge n]$$

is also arbitrarily close to 1, which in turn implies (by the triangle inequality, and equations (1) and (2)) that

$$P[|X_m + Y_m - X - Y| < \epsilon,\ \forall\, m \ge n] = P[|Z_m - Z| < \epsilon,\ \forall\, m \ge n]$$

is also arbitrarily close to 1, and hence $Z_n \xrightarrow{a.s.} Z$.

3. By definition

$$\lim_{n\longrightarrow\infty} E[|X_n - X|^2] = \lim_{n\longrightarrow\infty} E[(X_n - X)^2] = 0$$

But, for $n \leq m$,

$$|X_n - X|^2 = |(X_n - X_m + X_m - X)|^2 \leq |X_n - X_m|^2 + |X_m - X|^2$$

and

$$\lim_{n\longrightarrow\infty} E[|X_n - X|^2] = \lim_{m\longrightarrow\infty} E[|X_m - X|^2] = 0$$

so consequently

$$\lim_{n,m\longrightarrow\infty} E[|X_n - X_m|^2] = \lim_{n,m\longrightarrow\infty} E[(X_n - X_m)^2] = 0 \tag{3}$$

Now, for any two variables, U and V, we have

$$\{E[(UV)]\}^2 \leq E[U^2]E[V^2] \tag{4}$$

To see this, consider the variable $W = sU + V$; we have immediately that

$$0 \leq E[W^2] = E[(sU+V)^2] = E[s^2U^2 + 2sUV + V^2] = as^2 + bs + c.$$

where $a = E[U^2]$, $b = 2E[UV]$ and $c = E[V^2]$. Clearly $a \geq 0$, so consider $a > 0$ (if $a = 0$, then inequality (4) holds trivially). Then, as

$$g(s) = as^2 + bs + c$$

stays non-negative for all s, $g(s)$ has at most one real root. This implies that the "discriminant" is negative, that is

$$b^2 - \sqrt{4ac} \leq 0.$$

Consequently, substituting in the forms for a, b and c yields

$$(2E[UV])^2 - 4E[U^2]E[V^2] \leq 0$$

and the result in equation (4) follows.

Using equation (4), therefore,

$$\begin{aligned} Cov[X_n, X_m] &= E[(X_n - \mu)(X_m - \mu)] = E[(X_n - X_m + X_m - \mu)(X_m - \mu)] \\ &= E[(X_n - X_m)(X_m - \mu)] + E[(X_m - \mu)^2] \end{aligned}$$

But, by equation (4)

$$\{E[(X_n - X_m)(X_m - \mu)]\}^2 \leq E[(X_n - X_m)^2]E[(X_m - \mu)^2] = E[(X_n - X_m)^2]\sigma^2 \longrightarrow 0$$

as $n \longrightarrow \infty$, from equation (3). Hence

$$\begin{aligned} \lim_{n\longrightarrow\infty} Cov[X_n, X_m] &= \lim_{n\longrightarrow\infty} E[(X_n - X_m)(X_m - \mu)] + \lim_{n\longrightarrow\infty} E[(X_m - \mu)^2] \\ &= 0 + \sigma^2 \end{aligned}$$

and hence

$$Corr[X_n, X_m] = \frac{Cov[X_n, X_m]}{\sqrt{Var[X_n]Var[X_m]}} = \frac{Cov[X_n, X_m]}{\sqrt{\sigma^2\sigma^2}} \longrightarrow \frac{\sigma^2}{\sqrt{\sigma^2\sigma^2}} = 1$$

as $n \longrightarrow \infty$.

4. A result from lectures on almost sure convergence implies here that

$$I_n = \frac{1}{n}\sum_{i=1}^{n} g(U_i) \xrightarrow{a.s.} I \qquad \Longleftrightarrow \qquad E[|g(U)|] < \infty, \text{ with } I = E[g(U)]$$

so it is sufficient to check whether the function g is absolutely integrable on $(0,1)$. But

$$\int_0^1 |g(u)|du \quad = \quad \int_0^1 \left|\frac{1}{u}\sin(2\pi/u)\right| \, du = \int_0^1 \frac{1}{u}\left|\sin(2\pi/u)\right| \, du$$

and this integral is **unbounded**. To see this,

$$\begin{aligned}
E_{f_U}\left[\left|g\left(U\right)\right|\right] &= \int_0^1 \left|\frac{1}{u}\sin\left(\frac{2\pi}{u}\right)\right| du = \int_0^1 \frac{1}{u}\left|\sin\left(\frac{2\pi}{u}\right)\right| du \\
&= \int_1^\infty \frac{1}{y}\left|\sin\left(2\pi y\right)\right| dy \qquad \text{setting } y = 1/u. \\
&= \int_{2\pi}^\infty \frac{1}{t}\left|\sin t\right| dt \qquad \text{setting } t = 2\pi y. \\
&= \sum_{k=1}^{\infty}\left[\int_{2k\pi}^{(2k+1)\pi} \frac{1}{t}\sin t \, dt - \int_{(2k+1)\pi}^{2(k+1)\pi} \frac{1}{t}\sin t \, dt\right]
\end{aligned}$$

Now, in the first integral, on $(2k\pi, (2k+1)\pi)$, we have

$$\frac{1}{t} \geq \frac{1}{(2k+1)\pi}$$

and, in the second integral, on $((2k+1)\pi, 2\left(k+1\right)\pi)$, we have

$$\frac{1}{t} \leq \frac{1}{(2k+1)\pi}.$$

Hence

$$\begin{aligned}
E_{f_U}\left[\left|g\left(U\right)\right|\right] &\geq \sum_{k=1}^{\infty}\left[\int_{2k\pi}^{(2k+1)\pi} \frac{1}{(2k+1)\pi}\sin t \, dt - \int_{(2k+1)\pi}^{2(k+1)\pi} \frac{1}{(2k+1)\pi}\sin t \, dt\right] \\
&= \sum_{k=1}^{\infty}\frac{1}{(2k+1)\pi}\left[\int_{2k\pi}^{(2k+1)\pi} \sin t \, dt - \int_{(2k+1)\pi}^{2(k+1)\pi} \sin t \, dt\right] \\
&= \sum_{k=1}^{\infty}\frac{1}{(2k+1)\pi}\left[\left[-\cos t\right]_{2k\pi}^{(2k+1)\pi} - \left[-\cos t\right]_{(2k+1)\pi}^{2(k+1)\pi}\right] \\
&= \sum_{k=1}^{\infty}\frac{1}{(2k+1)\pi}\left[2-(-2)\right] \\
&= \sum_{k=1}^{\infty}\frac{4}{(2k+1)\pi}
\end{aligned}$$

and the final sum is divergent.

5. By definition, if $i = \sqrt{-1}$, then

$$C_{\boldsymbol{X}}(\boldsymbol{t}) = E_{f_{\boldsymbol{X}}}[\exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{X}\}] = \int \exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{X}\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x}$$

where the final integral is k-dimensional. Partially differentiating with respect to t_j of this form yields

$$\begin{aligned} \frac{\partial}{\partial t_j}\left\{\int \exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x}\right\} &= \int \frac{\partial}{\partial t_j}\left\{\exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\}\right\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int i x_j \left\{\exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\}\right\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \end{aligned}$$

which when evaluated at $\boldsymbol{t} = 0$, yields

$$\int i x_j f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \equiv i\mu_j.$$

Repeating for each $j = 1, \ldots, k$ yields the result.

Similarly,

$$\begin{aligned} \frac{\partial^2}{\partial t_j \partial t_l}\left\{\int \exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x}\right\} &= \int \frac{\partial^2}{\partial t_j \partial t_l}\left\{\exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\}\right\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int (i x_j)(i x_l) \left\{\exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{x}\}\right\} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \end{aligned}$$

which when evaluated at $\boldsymbol{t} = 0$, yields

$$\int -1 x_j x_l f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \equiv -E_{f_{X_j,X_l}}[X_j X_l]$$

as $i \times i = -1$. Forming the $k \times k$ matrix of such expectations derived from partial derivatives yields the result, as

$$\boldsymbol{X}\boldsymbol{X}^{\mathsf{T}} = [X_j X_l]_{j,l=1,\ldots,k}$$