M3S3/M4S3 : SOLUTIONS 1

1. First let $E_1 \subset E_2 \subset E_3 \subset \ldots$ is an increasing sequence of sets. Using the decomposition given,

$$E_{n+1} \equiv E_n \cup \left(E_{n+1} \cap E_n'\right)$$

it follows that

$$\lim_{n\to\infty} E_n \equiv \bigcup_{n=1}^{\infty} E_n \equiv E_1 \cup \bigcup_{n=2}^{\infty} F_n$$

where

$$F_n \equiv \left(E_n \cap E_{n-1}'\right).$$

But, the $\{F_n\}$ are a sequence of mutually exclusive events, so by the probability axioms

$$P\left(\lim_{n\to\infty} E_n\right) = P\left(E_1 \cup \bigcup_{n=2}^{\infty} F_n\right) = P(E_1) + \sum_{n=2}^{\infty} P(F_n) = P(E_1) + \lim_{n\to\infty} \sum_{k=2}^{n} P(F_k).$$

Now, for this increasing sequence,

$$P(F_k) = P(E_k \cap E_{k-1}') = P(E_k) - P(E_{k-1})$$

and hence

$$P\left(\lim_{n\to\infty} E_n\right) = P(E_1) + \lim_{n\to\infty} \sum_{k=2}^{n} (P(E_k) - P(E_{k-1})) = \lim_{n\to\infty} P(E_n),$$

as the sum telescopes, successive terms cancelling. Thus

$$P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Now, if $\{E_n\}$ is *decreasing*, then the sequence $\{G_n\}$ defined by $G_n \equiv E_n'$ is *increasing*, so by above

$$P\left(\lim_{n\to\infty} G_n\right) = \lim_{n\to\infty} P(G_n).$$

But, by De Morgan's Law

$$\lim_{n\to\infty} G_n \equiv \bigcup_{n=1}^{\infty} G_n \equiv \bigcup_{n=1}^{\infty} E_n' \equiv \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

so

$$P\left(\lim_{n\to\infty} G_n\right) = P\left(\left(\bigcap_{n=1}^{\infty} E_n\right)'\right) = 1 - P\left(\bigcap_{n=1}^{\infty} E_n\right) = 1 - P\left(\lim_{n\to\infty} E_n\right),$$

and also

$$\lim_{n\to\infty} P(G_n) = \lim_{n\to\infty} P(E_n') = \lim_{n\to\infty} (1 - P(E_n)) = 1 - \lim_{n\to\infty} P(E_n).$$

Thus, substituting back into the previous relationship

$$1 - P\left(\lim_{n\to\infty} E_n\right) = 1 - \lim_{n\to\infty} P(E_n)$$

and hence

$$P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

2. (a) Denote by E^∞ the set of $\omega \in \Omega$ such that $f(\omega) = \infty$. As f is measurable, $E^\infty \in \mathcal{F}$, and by definition $A \equiv E \cap E^\infty \in \mathcal{F}$ and thus is also measurable. Now, define simple function ψ_n by

$$\psi_n(\omega) = nI_A(\omega).$$

Then, for $n = 1, 2, 3, \ldots$,

$$0 \leq \psi_n \leq f$$

as if $\omega \in A$, then $f(\omega) = \infty$, and if $\omega \notin A$, then $\psi_n(\omega) = 0$. Now, for $n \geq 1$,

$$\int_E \psi_n \, d\nu = \int_\Omega I_E \psi_n \, d\nu = n \int_\Omega I_E I_A \, d\nu = n\nu(E \cap A) = n\nu(A) \tag{1}$$

by apply the definition of the integral for simple functions, and as $A \equiv E \cap E^\infty$. But, by the supremum definition,

$$\int_E f \, d\nu = \sup_{\psi \,:\, 0 \leq \psi \leq f} \left\{ \int_E \psi \, d\nu \right\} \geq \int_E \psi_n \, d\nu$$

for our *specifically* defined ψ_n, as $0 \leq \psi_n \leq f$. Thus by assumption, and the previous result in equation (1), we have

$$n\nu(A) = \int_E \psi_n \, d\nu \leq \int_E f \, d\nu < \infty.$$

But this must hold for all $n \geq 1$, so it must follow that $\nu(A) = 0$, as if $\nu(A) > 0$ then $n\nu(A)$ is unbounded.

(b) Consider simple function $\psi \leq f$ (at least on E). Then, by construction $\psi \leq MI_E$ also, so

$$\int_E \psi \, d\nu \leq \int_E MI_E \, d\nu = M\nu(E),$$

that is, $M\nu(E)$ is **an** upper bound on the integral of ψ. But, by the supremum definition, it follows that

$$\int_E f \, d\nu \leq M\nu(E) \tag{2}$$

as the integral on the left-hand side is the supremum, or **least** upper bound.

Now define simple function $\psi_m = mI_E$, so that $\psi_m \leq f$, so, by construction,

$$\int_E \psi_m \, d\nu = m\nu(E).$$

But, by the supremum definition, as $\psi_m \leq f$,

$$m\nu(E) = \int_E \psi_m \, d\nu \leq \int_E f \, d\nu \tag{3}$$

as the final integral is an upper bound on integrals of simple functions ψ such that $\psi \leq f$. Hence

$$m\nu(E) \leq \int_E f \, d\nu \leq M\nu(E)$$

by equations (2) and (3).

(c) Suppose first that f is non-negative. Consider arbitrary simple function ψ of the general form

$$\psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega).$$

such that $0 \leq \psi \leq f$. Then by definition

$$\int_E \psi \, d\nu = \sum_{i=1}^{k} a_i \nu(E \cap A_i) = 0$$

as $\nu(E \cap A_i) \leq \nu(E) = 0$ by assumption. Therefore it also follows by the supremum definition that

$$\int_E f \, d\nu = 0$$

as zero is an upper bound, and the integral of f is the **least** upper bound.

For f negative, apply the argument to $-f$ and the result follows in the same fashion.

(d) For $n \geq 1$, define set B_n by

$$B_n \equiv \left\{ \omega : \ \omega \in E, f(\omega) > \frac{1}{n} \right\}.$$

Then $\{B_n\}$ is an increasing sequence of sets and so

$$\lim_{n \to \infty} B_n \equiv \bigcup_{n=1}^{\infty} \equiv B.$$

But for each n,

$$\frac{1}{n}\nu(B_n) = \frac{1}{n}\int_E I_{B_n} \, d\nu = \int_E \frac{1}{n} I_{B_n} \, d\nu \leq \int_E f \, d\nu \tag{4}$$

as $f \geq 1/n$ on B_n, and also $B_n \subseteq E$. But, by assumption, the final integral in equation (4) is zero, so $\nu(B_n) = 0$. Hence, by the continuity of measure (proved in Q1)

$$\nu(B) = \nu\left(\lim_{n \to \infty} B_n\right) = \lim_{n \to \infty} \nu(B_n) = 0$$

and B has measure zero under ν.

(e) Let ψ be a simple function satisfying $0 \leq \psi \leq f$, say

$$\psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega).$$

Then

$$\int_{E_1} \psi \, d\nu \leq \int_{E_2} \psi \, d\nu \tag{5}$$

as

$$E_1 \subseteq E_2 \Rightarrow (E_1 \cap A_i) \subseteq (E_2 \cap A_i) \Rightarrow \nu(E_1 \cap A_i) \leq \nu(E_2 \cap A_i)$$

for each i in the definition of ψ. Taking supremum on the left and right-hand sides of equation (5) preserves the inequality, and yields the result, by the supremum definition of the integral of f.

(f) Let D be the set of $\omega \in E$ where $f(\omega) > g(\omega)$, so that $\nu(D) = 0$. Consider simple function ψ with $0 \leq \psi \leq f$, and function $\psi^\star$ defined by

$$\psi^\star = \psi I_{D'}.$$

By construction, $\psi^\star$ is a simple function, and $0 \leq \psi^\star \leq g$, and thus by the supremum definition,

$$\int_E \psi \, d\nu = \int_E \psi^\star \, d\nu \leq \int_E g \, d\nu$$

and the integral of g is **an** upper bound on the integral of ψ, where $0 \leq \psi \leq f$. But, by the supremum definition, the integral of f is the **least** upper bound of integrals of such ψ functions, so it must follow that

$$\int_E f \, d\nu \leq \int_E g \, d\nu.$$