M3S3/S4 STATISTICAL THEORY II

POSITIVE DEFINITE MATRICES

Definition: Positive Definite Matrix
A square, $p \times p$ symmetric matrix A is *positive definite* if, for all $x \in \mathbb{R}^p$,

$$x^\mathsf{T} A x > 0$$

Properties: Suppose that A

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix}$$

is a positive definite matrix.

1. The $r \times r$ $(1 \le r \le p)$ submatrix A_r,

$$A_r = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix}$$

   is also positive definite.

2. The p eigenvalues of A, $\lambda_1, \ldots, \lambda_p$ are **positive**. Conversely, if all the eigenvalues of a matrix B are positive, then B is positive definite.

3. There exists a unique decomposition of A

$$A = LL^\mathsf{T} \tag{1}$$

   where L is a lower triangular matrix

$$L = [l_{ij}] = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{p1} & l_{p2} & \cdots & l_{pp} \end{bmatrix}.$$

   Equation (1) gives the *Cholesky Decomposition* of A.

4. There exists a unique decomposition of A

$$A = SS \tag{2}$$

   where S can be denoted $A^{1/2}$. S is the *matrix square root* of A.

5. There exists a unique decomposition of A

$$A = VDV^\mathsf{T} \tag{3}$$

where

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{bmatrix}$$

is the diagonal matrix composed of the eigenvalues of A, and V is an *orthogonal matrix*

$$V^{\mathsf{T}} V = \mathbf{1}$$

Equation (3) gives the *Singular Value Decomposition* of A.

6. As $A = VDV^{\mathsf{T}}$,

$$|A| = |VDV^{\mathsf{T}}| = |V||D||V^{\mathsf{T}}| = |V|^2|D| = |D| > 0$$

as

$$|V| = 1 \qquad \text{and} \qquad |D| = \prod_{i=1}^{p} \lambda_i > 0$$

by 2 and 5.

7. By 6., as $|A| > 0$, A is *non-singular*, that is, the *inverse* of A, A^{-1} exists such that

$$AA^{-1} = A^{-1}A = \mathbf{1}.$$

In fact

$$A^{-1} = (VDV^{\mathsf{T}})^{-1} = VD^{-1}V^{\mathsf{T}}$$

as

$$V^{-1} = V^{\mathsf{T}}.$$

8. A^{-1} is positive definite.

9. For $x \in \mathbb{R}^p$,

$$\min_{1 \le i \le p} \lambda_i \le \frac{x^{\mathsf{T}} A x}{x^{\mathsf{T}} x} \le \max_{1 \le i \le p} \lambda_i$$

10. If A and B are positive definite, then

    (i) $|A + B| \le |A| + |B|$.
    (ii) If $A - B$ is positive definite, $|A| > |B|$.
    (iii) $B^{-1} - A^{-1}$ is positive definite.