M3/M4S3 STATISTICAL THEORY II

LAWS OF LARGE NUMBERS

Theorem 1.9 ***Behaviour of the Sample Mean.***
Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots,$ be independent, identically distributed (i.i.d.) random variables in $\mathbb{R}^k$. Let

$$\overline{\boldsymbol{X}}_n = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{X}_j.$$

(a) ***The Weak Law*** *If $E[|\boldsymbol{X}|] < \infty$, then*

$$\overline{\boldsymbol{X}}_n \xrightarrow{p} \boldsymbol{\mu} = E[\boldsymbol{X}]$$

(b) If $E[|\boldsymbol{X}|^2] < \infty$, then

$$\overline{\boldsymbol{X}}_n \xrightarrow{r=2} \boldsymbol{\mu} = E[\boldsymbol{X}]$$

also written

$$\overline{\boldsymbol{X}}_n \xrightarrow{q.m.} \boldsymbol{\mu} = E[\boldsymbol{X}]$$

where q.m. stands for "quadratic mean".

(c) ***The Strong Law***

$$\overline{\boldsymbol{X}}_n \xrightarrow{a.s.} \boldsymbol{\mu} \qquad \Longleftrightarrow \qquad E[|\boldsymbol{X}|] < \infty \text{ and } E[\boldsymbol{X}] = \boldsymbol{\mu}$$

Proof. (a) Proof uses characteristic functions (cfs): let the cf of $\boldsymbol{X}$ be $C_{\boldsymbol{X}}(\boldsymbol{t})$. Then, by elementary generating function results for sums of i.i.d. variables,

$$C_{\overline{\boldsymbol{X}}_n}(\boldsymbol{t}) \quad = \quad \{C_{\boldsymbol{X}}(\boldsymbol{t}/n)\}^n$$

and by the Mean-Value Theorem with $\boldsymbol{x_0} = \boldsymbol{0}$,

$$\begin{aligned} C_{\overline{\boldsymbol{X}}_n}(\boldsymbol{t}) &= \left\{ C_{\boldsymbol{X}}(\boldsymbol{0}) + \left[\int_0^1 \dot{C}_{\overline{\boldsymbol{X}}_n}(u\boldsymbol{t}/n)\, du\right] \frac{\boldsymbol{t}}{n} \right\}^n \\ &= \left\{ 1 + \left[\int_0^1 \dot{C}_{\overline{\boldsymbol{X}}_n}(u\boldsymbol{t}/n)\, du\right] \frac{\boldsymbol{t}}{n} \right\}^n. \end{aligned}$$

But, as $n \longrightarrow \infty$, $u\boldsymbol{t}/n \longrightarrow \boldsymbol{0}$, so

$$\dot{C}_{\overline{\boldsymbol{X}}_n}(u\boldsymbol{t}/n) \longrightarrow \dot{C}_{\overline{\boldsymbol{X}}_n}(\boldsymbol{0}) = i\boldsymbol{\mu}^{\mathsf{T}}.$$

Thus,

$$C_{\overline{\boldsymbol{X}}_n}(\boldsymbol{t}) \longrightarrow \left\{1 + \frac{i\boldsymbol{\mu}^{\mathsf{T}}\boldsymbol{t}}{n}\right\}^n \longrightarrow \exp\{i\boldsymbol{\mu}^{\mathsf{T}}\boldsymbol{t}\} = \exp\{i\boldsymbol{t}^{\mathsf{T}}\boldsymbol{\mu}\}$$

as $n \longrightarrow \infty$. But this is just the cf of a random variable that is degenerate at $\boldsymbol{\mu}$, so the result follows.

(b) We have

$$
\begin{aligned}
E[|\overline{\boldsymbol{X}}_{\boldsymbol{n}} - \boldsymbol{\mu}|^2] &= E[(\overline{\boldsymbol{X}}_{\boldsymbol{n}} - \boldsymbol{\mu})^\mathsf{T}(\overline{\boldsymbol{X}}_{\boldsymbol{n}} - \boldsymbol{\mu})] = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l=1}^{n} E[(\boldsymbol{X}_j - \boldsymbol{\mu})^\mathsf{T}(\boldsymbol{X}_l - \boldsymbol{\mu})] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} E[(\boldsymbol{X}_j - \boldsymbol{\mu})^\mathsf{T}(\boldsymbol{X}_j - \boldsymbol{\mu})] \qquad \text{by independence} \\
&= \frac{1}{n} E[(\boldsymbol{X} - \boldsymbol{\mu})^\mathsf{T}(\boldsymbol{X} - \boldsymbol{\mu})] \qquad \text{as all terms in sum are identical constants} \\
&\longrightarrow 0
\end{aligned}
$$

as $n \longrightarrow \infty$. Thus $\overline{\boldsymbol{X}}_{\boldsymbol{n}} \xrightarrow{r=2} \boldsymbol{\mu}$.

(c) Omitted.