M3S3/M4S3 - EXERCISES 2

MODES OF CONVERGENCE

1. For the following sequences of random variables, $\{X_n\}$, decide whether the the sequence converges *almost surely*, or in *mean-square* (rth mean for $r = 2$), or *in probability* as $n \longrightarrow \infty$.

(a) $X_n = \begin{cases} 1 & \text{with prob. } 1/n \\ 2 & \text{with prob. } 1 - 1/n \end{cases}$

(b) $X_n = \begin{cases} n^2 & \text{with prob. } 1/n \\ 1 & \text{with prob. } 1 - 1/n \end{cases}$

(c) $X_n = \begin{cases} n & \text{with prob. } 1/\log n \\ 0 & \text{with prob. } 1 - 1/\log n \end{cases}$

2. Suppose that, for sequences of random variables, $\{X_n\}$ and $\{Y_n\}$,

$$X_n \stackrel{r=2}{\longrightarrow} X \qquad \text{and} \qquad Y_n \stackrel{r=2}{\longrightarrow} Y$$

as $n \longrightarrow \infty$. Prove that

$$Z_n = X_n + Y_n \stackrel{r=2}{\longrightarrow} Z = X + Y$$

as $n \longrightarrow \infty$. *Hint: Recall the Cauchy-Schwarz Inequality.*

Does the result hold if you replace convergence in $r = 2$ mean by convergence in probability or convergence almost surely ? Justify your answer.

3. Suppose $X_n \stackrel{r=2}{\longrightarrow} X$ as $n \longrightarrow \infty$. Show that, for $n \leq m$,

$$E[(X_n - X_m)^2] \longrightarrow 0$$

as $n, m \longrightarrow \infty$. If $E[X_n] = \mu$ and $Var[X_n] = \sigma^2 < \infty$ for all n, find the limiting value of the correlation

$$Corr[X_n, X_m]$$

as $n, m \longrightarrow \infty$.

4. An estimator of the integral

$$I = \int_1^\infty \frac{\sin(2\pi x)}{x}\, dx = \int_0^1 \frac{\sin(2\pi/u)}{u}\, du = \int_0^1 g(u)\, du$$

say, can be constructed using so-called *Monte Carlo* methods as

$$I_n = \frac{1}{n} \sum_{i=1}^n \frac{1}{U_i} \sin\left(\frac{2\pi}{U_i}\right) = \frac{1}{n} \sum_{i=1}^n g(U_i)$$

where $U_1, \ldots, U_n \sim Uniform(0, 1)$ are i.i.d. variables. The true value of I is 0.1526447507 (from `MAPLE`).

Does $I_n \stackrel{a.s.}{\longrightarrow} I$? Justify your answer.

5. Prove the results given in lectures relating to characteristic functions, namely

$$\dot{C}_{\boldsymbol{X}}(\mathbf{0}) = i\mu^\mathsf{T} \qquad \ddot{C}_{\boldsymbol{X}}(\mathbf{0}) = -E[\boldsymbol{X}\boldsymbol{X}^\mathsf{T}]$$

when these quantities are finite.