M3S3/M4S3
ASSESSED COURSEWORK 1

Deadline: Friday 24th February
Please hand in during Lecture/to Room 523

1. Suppose that $X_1, \ldots, X_n$ are an i.i.d. sample from a Normal distribution with expectation μ and variance σ^2. Find the asymptotic distribution of

(i) the **sample median**, $X_{(k)}$, that is the $p = 0.5$ sample quantile, so that $k = \lceil np \rceil$ with $p = 0.5$.

[5 MARKS]

(ii) the **sample interquartile range**, R_{IQ}, defined by

$$R_{IQ} = X_{(k_2)} - X_{(k_1)}$$

with $k_1 = \lceil np_1 \rceil$ and $k_2 = \lceil np_2 \rceil$ with $p_1 = 0.25$ and $p_2 = 0.75$, that is, the difference between the 0.75 sample quantile and the 0.25 sample quantile.

[10 MARKS]

Use the following results; if $\Phi(.)$ is the standard normal cdf, then

$$\Phi(-0.674) = 0.25 \qquad \Phi(0.674) = 0.75.$$

Recall that the standard normal density takes the form

$$f_X(x) = \left(\frac{1}{2\pi}\right)^{1/2} \exp\{-x^2/2\} \qquad x \in \mathbb{R},$$

and that if $Z = (Z_1, Z_2)^\top$ has a bivariate normal distribution

$$Z \sim N(\boldsymbol{\mu}, \Sigma) \qquad \text{with} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

then

$$Z_1 \sim N(\mu_1, \sigma_1^2) \qquad \text{and} \qquad \boldsymbol{a}^\top Z \sim N(\boldsymbol{a}^\top \boldsymbol{\mu}, \boldsymbol{a}^\top \Sigma \boldsymbol{a})$$

for vector $\boldsymbol{a}$, a (2×1) constant vector.

2. The (squared)[1] Hellinger distance, d_H, between two univariate densities f_1 and f_2 (defined with respect to Lebesgue measure) can be written

$$d_H(f_1, f_2) = \int_{-\infty}^{\infty} \left(\sqrt{f_1(x)} - \sqrt{f_2(x)}\right)^2 \, dx. \qquad (1)$$

Find an upper bound for $d_H(f_1, f_2)$ that holds for arbitrary f_1, f_2.

[5 MARKS]

Note: as usual, $\sqrt{.}$ indicates **positive** square root.

[1] some texts refer to (1) as the squared Hellinger distance , and use the notation $d_H^2(f_1, f_2)$.