SAMPLE EXAM QUESTION 1

(a) State Cramer's result (also known as the Delta Method) on the asymptotic normal distribution of a (scalar) random variable Y defined in terms of random variable X via the transformation $Y = g(X)$, where X is asymptotically normally distributed

$$X \sim AN\left(\mu, \sigma^2\right).$$

[4 MARKS]

(b) Suppose that $X_1, ..., X_n$ are independent and identically distributed $Poisson\,(\lambda)$ random variables. Find the maximum likelihood (ML) estimator, and an asymptotic normal distribution for the estimator, of the following parameters

(i) λ,
(ii) $\exp\{-\lambda\}$.

[6 MARKS]

(c) Suppose that, rather than observing the random variables in (b) precisely, only the events

$$X_i = 0 \qquad \text{or} \qquad X_i > 0$$

for $i = 1, ..., n$ are observed.

(i) Find the ML estimator of λ under this new observation scheme.
(ii) In this new scheme, when does the ML estimator not exist (at a finite value in the parameter space) ? Justify your answer.
(iii) Compute the probability that the ML estimator does not exist for a finite sample of size n, assuming that the true value of λ is λ_0.
(iv) Construct a modified estimator that is consistent for λ.

[2,2,3,3 MARKS]

SAMPLE EXAM QUESTION 2

(a) Suppose that $X_{(1)} < \ldots < X_{(n)}$ are the order statistics from a random sample of size n from a distribution F_X with continuous density f_X on $\mathbb{R}$. Suppose $0 < p_1 < p_2 < 1$, and denote the quantiles of F_X corresponding to p_1 and p_2 by x_{p_1} and x_{p_2} respectively.

Regarding x_{p_1} and x_{p_2} as unknown parameters, natural estimators of these quantities are $X_{(\lceil np_1 \rceil)}$ and $X_{(\lceil np_2 \rceil)}$ respectively, where $\lceil x \rceil$ is the smallest integer not less than x. Show that

$$\sqrt{n}\begin{pmatrix} X_{(\lceil np_1 \rceil)} - x_{p_1} \\ X_{(\lceil np_2 \rceil)} - x_{p_2} \end{pmatrix} \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

where

$$\Sigma = \begin{bmatrix} \dfrac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2} & \dfrac{p_1(1-p_2)}{f_X(x_{p_1})f_X(x_{p_2})} \\ \\ \dfrac{p_1(1-p_2)}{f_X(x_{p_1})f_X(x_{p_2})} & \dfrac{p_2(1-p_2)}{\{f_X(x_{p_2})\}^2} \end{bmatrix}$$

State the equivalent result for a single quantile x_p corresponding to probability p.

[10 MARKS]

(b) Using the results in (a), find the asymptotic distribution of

(i) The sample median estimator of the median F_X (corresponding to $p = 0.5$), if F_X is a Normal distribution with parameters μ and σ^2.

(ii) The upper and lower quartile estimators (corresponding to $p_1 = 0.25$ and $p_2 = 0.75$) if F_X is an Exponential distribution with parameter λ

[3,3 MARKS]

(c) The results in (a) and (b) describe convergence in law for the estimators concerned. Show how the form of convergence may be strengthened using the Strong Law for any specific quantile x_p.

[4 MARKS]

SAMPLE EXAM QUESTION 3

(a) (i) State (without proof) Wald's Theorem on the strong consistency of maximum likelihood (ML) estimators, listing the five conditions under which this theorem holds.

[5 MARKS]

(ii) Verify that the conditions of the theorem hold when random variables $X_1, \ldots, X_n$ correspond to independent observations from the Uniform density on $(0, \theta)$

$$f_X(x|\theta) = \frac{1}{\theta} \qquad 0 \leq x \leq \theta$$

and zero otherwise, for parameter $\theta \in \Theta \equiv [a, b]$, where $[a, b]$ is the closed interval from a to b, $0 < a < b < \infty$.

[Hint: for $x \in \mathbb{R}$, consider the function

$$K(x) = \max_{\theta \in \Theta} \frac{f_X(x|\theta)}{f_X(x|\theta_0)}$$

where $\theta_0 \in \Theta$ is the true value of θ.]

[5 MARKS]

(b) Wald's Theorem relates to one form of consistency; the remainder of the question focuses on another form.

Suppose that random variables $X_1, \ldots, X_n$ correspond independent observations from density (wrt Lebesgue measure) $f_{X|\theta}(x|\theta)$, and for $\theta \in \Theta$, this family of densities have common support $\mathbb{X}$. Let the true value of θ be denoted θ_0, and let $L_n(\theta)$ denote the likelihood for θ

$$L_n(\theta) = \prod_{i=1}^{n} f_{X|\theta}(x_i|\theta).$$

(i) Using Jensen's inequality for the function $g(x) = -\log x$, and an appropriate law of large numbers, show that

$$P_{\theta_0}\left[L_n(\theta_0) > L_n(\theta)\right] \longrightarrow 1 \qquad \text{as} \qquad n \longrightarrow \infty$$

for any **fixed** $\theta \neq \theta_0$, where P_{θ_0} denotes probability under the true model, indexed by θ_0.

Which other condition from (a)(i) needs to be assumed in order for the result to hold ?

[5 MARKS]

(ii) Suppose that, in addition to the conditions listed in (b), parameter space Θ is finite, that is, $\Theta \equiv \{t_1, \ldots, t_p\}$ for some positive integer p.

Show that, in this case, the ML estimator $\widehat{\theta}_n$ exists, and is weakly consistent for θ_0.

[5 MARKS]

This one's a bit harder, but worth working through. First bit is bookwork. DAS.

SAMPLE EXAM QUESTION 4

(a) (i) Give definitions for the following modes of stochastic convergence, summarizing the relationships between the various modes;

- convergence in law (convergence in distribution)
- convergence almost surely
- convergence in r^{th} mean

[6 MARKS]

(ii) Consider the sequence of random variables $X_1, X_2, \ldots$ defined by

$$X_n(Z) = nI_{[0,n)}(Z)$$

where Z is a single random variable having an *Exponential* distribution with parameter 1.

Under which modes of convergence does the sequence $\{X_n\}$ converge ? Justify your answer.

[5 MARKS]

(b) Suppose that $X_1, X_2, \ldots$ are independent, identically distributed random variables defined on $\mathbb{R}$, with common distribution function F_X for which $F_X(x) < 1$ for all finite x. Let M_n be the maximum random variable defined for finite n by

$$M_n = \max\{X_1, X_2, \ldots, X_n\}$$

Hint: use the Borel-Cantelli lemma.

(i) Show that the sequence of random variables $\{M_n\}$ converges almost surely to infinity, that is

$$M_n \stackrel{a.s.}{\longrightarrow} \infty$$

as $n \to \infty$.

[5 MARKS]

(ii) Now suppose that $F_X(x_U) = 1$ for some $x_U < \infty$. Find the almost sure limiting random variable for the sequence $\{M_n\}$.

[5 MARKS]

SAMPLE EXAM QUESTION 5

(a) Suppose that $X_1, \ldots, X_n$ are an independent and identically distributed sample from distribution with density $f_{X|\theta}(x|\theta)$, for vector parameter $\theta \in \Theta \subseteq \mathbb{R}^d$. Suppose that $f_{X|\theta}$ is twice differentiable with respect to the elements of θ, and let the true value of θ be denoted θ_0.

Define

(i) The *Score Statistic* (or Score function), $S(X;\theta)$.

(ii) The *Fisher Information* for a single random variable, $I(\theta)$

(iii) The *Fisher Information* for the sample of size n, $I_n(\theta)$.

(iv) The *Estimated Fisher Information*, $\widehat{I}_n(\theta)$.

Give the asymptotic Normal distribution of the score statistic under standard regularity conditions, when the data are distributed as a Normal distribution with mean zero and variance $1/\theta$.

[10 MARKS]

(b) One class of estimating procedures for parameter θ involves solution of equations of the form

$$G_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} G_i(X_i;\theta) = 0 \tag{1}$$

for suitably defined functions $G_i, i = 1, \ldots, n$.

(i) Show that maximum likelihood (ML) estimation falls into this class of estimating procedures.

(ii) Suppose that $\hat{\theta}_n$ is a solution to (1) which is weakly consistent for θ.

Using an approximation to G_n (motivated by a Taylor expansion of G_n around θ_0) of the form

$$G_n(\hat{\theta}_n) = G_n(\theta_0) + (\widehat{\theta}_n - \theta_0)\dot{G}_n(\theta_0),$$

where $\dot{G}_n$ is the first partial derivative vector wrt the d components of θ, find an asymptotic normal distribution of $\widehat{\theta}_n$.

State precisely the assumptions made in order to obtain the asymptotic Normal distribution.

[10 MARKS]

SAMPLE EXAM QUESTION 6

(a) Suppose that $X_1, \ldots, X_n$ are a finitely exchangeable sequence of random variables with (De Finetti) representation

$$p(X_1, \ldots, X_n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} f_{X|\theta}(X_i|\theta) p_\theta(\theta) d\theta$$

In the following cases, find the joint probability distribution $p(X_1, \ldots, X_n)$, and give an interpretation of the parameter θ in terms of a strong law limiting quantity.

(i)

$$\begin{aligned} f_{X|\theta}(X_i|\theta) &= Normal(\theta, 1) \\ p_\theta(\theta) &= Normal(0, \tau^2) \end{aligned}$$

for parameter $\tau > 0$.

(ii)

$$\begin{aligned} f_{X|\theta}(X_i|\theta) &= Exponential(\theta) \\ p_\theta(\theta) &= Gamma(\alpha, \beta) \end{aligned}$$

for parameters $\alpha, \beta > 0$.

[5 MARKS each]

(b) In each of the two cases of part (a), compute the posterior predictive distribution

$$p(X_{m+1}, \ldots, X_{m+n}|X_1, \ldots, X_m)$$

for $0 < n, m$ where $X_1, \ldots, X_{m+n}$ are a finitely exchangeable sequence

Find in each case the limiting posterior predictive distribution as $n \longrightarrow \infty$.

[5 MARKS each]