M3S12 BIOSTATISTICS - EXERCISES 3
SOLUTIONS

1.(a) Under the normal model, the likelihood is

$$L(\mu,\sigma) = f_{X|\mu,\sigma}\left(x;\mu,\sigma^2\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

and thus, in terms of the random variables, for X_1,

$$\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right) = -\frac{1}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}(X_1-\mu)^2$$

and, for μ

$$\frac{\partial}{\partial\mu}\left\{\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right)\right\} = \frac{1}{\sigma^2}(X_1-\mu) \qquad (1)$$

$$\frac{\partial^2}{\partial\mu^2}\left\{\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right)\right\} = -\frac{1}{\sigma^2}$$

whereas for σ^2

$$\frac{\partial}{\partial\sigma^2}\left\{\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right)\right\} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(X_1-\mu)^2 \qquad (2)$$

$$\frac{\partial^2}{\partial\left(\sigma^2\right)^2}\left\{\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right)\right\} = \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}(X_1-\mu)^2$$

and

$$\frac{\partial^2}{\partial\mu\partial\sigma^2}\left\{\log f_{X_1|\mu,\sigma}\left(X_1;\mu,\sigma^2\right)\right\} = -\frac{1}{\sigma^4}(X_1-\mu)$$

(here taking σ^2 as the variable with which we differentiating with respect to). Now because

$$E_{f_{X_1|\mu,\sigma}}\left[(X_1-\mu)\right] = 0 \qquad E_{f_{X_1|\mu,\sigma}}\left[(X_1-\mu)^2\right] = \sigma^2$$

we have for the Fisher Information for $\left(\mu,\sigma^2\right)$ from a single datum as

$$I_1\left(\mu,\sigma^2\right) = -\begin{bmatrix} E\left[-\frac{1}{\sigma^2}\right] & E\left[-\frac{1}{\sigma^4}(X_1-\mu)\right] \\ E\left[-\frac{1}{\sigma^4}(X_1-\mu)\right] & E\left[\frac{1}{2\sigma^4} - \frac{1}{\sigma^6}(X_1-\mu)^2\right] \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

and thus

$$I_n\left(\mu,\sigma^2\right) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix}$$

and

$$I_1^{-1}\left(\mu,\sigma^2\right) = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix} \qquad I_n^{-1}\left(\mu,\sigma^2\right) = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

(b) (i) The **LR** statistic is

$$T_n = \frac{\sup\limits_{\theta \in \Theta_1} f_{X|\theta}(X;\theta)}{\sup\limits_{\theta \in \Theta_0} f_{X|\theta}(X;\theta)}$$

Under H_0, the MLE of σ^2 is

$$S_0^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

whereas under H_1, the MLEs of μ and σ^2 are (by the usual arguments)

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2.$$

Thus

$$T_n = \frac{L\left(\overline{X}, S^2\right)}{L\left(0, S_0^2\right)} = \frac{\left(\dfrac{1}{2\pi S^2}\right)^{n/2} \exp\left\{-\dfrac{1}{2S^2}\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right\}}{\left(\dfrac{1}{2\pi S_0^2}\right)^{n/2} \exp\left\{-\dfrac{1}{2S_0^2}\displaystyle\sum_{i=1}^{n} X_i{}^2\right\}} = \left(\frac{S_0^2}{S^2}\right)^{n/2}$$

and

$$2\log T_n = n\log\left(\frac{S_0^2}{S^2}\right).$$

But, also,

$$\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\overline{X}\right)^2$$

so

$$S^2 = S_0^2 - \left(\overline{X}\right)^2$$

and thus

$$2\log T_n = n\log\left(\frac{S_0^2}{S_0^2 - \left(\overline{X}\right)^2}\right) = -n\log\left(1 - \frac{\left(\overline{X}\right)^2}{S_0^2}\right). \tag{3}$$

(ii) The **Rao** Statistic is

$$R_n = Z_{n0}^T \left[\hat{I}_n\left(\widetilde{\theta}_n^{(0)}\right)\right]^{-1} Z_{n0}$$

where $\widetilde{\theta}_n^{(0)}$ is the estimate of θ under H_0. Now, using an estimate based on the MLE of σ **under** H_0

$$\left[\hat{I}_n\left(\widetilde{\theta}_n^{(0)}\right)\right]^{-1} = \begin{bmatrix} \widehat{\sigma}^2 & 0 \\ 0 & 2\widehat{\sigma}^4 \end{bmatrix} = \begin{bmatrix} S_0^2 & 0 \\ 0 & 2S_0^4 \end{bmatrix}$$

and, as $\mu = 0$ under H_0, from (1) and (2),

$$Z_{n0} = \frac{1}{\sqrt{n}} l_n' \left(\widetilde{\theta}_n^{(0)}\right) = \frac{1}{\sqrt{n}} \begin{bmatrix} \dfrac{1}{\widehat{\sigma}^2} \sum\limits_{i=1}^n X_i \\ -\dfrac{1}{2\widehat{\sigma}^2} + \dfrac{1}{2\widehat{\sigma}^4} \sum\limits_{i=1}^n X_i{}^2 \end{bmatrix} = \begin{bmatrix} \sqrt{n}\overline{X}/S_0^2 \\ 0 \end{bmatrix}$$

and thus

$$Z_{n0} = \begin{bmatrix} \sqrt{n}\overline{X}/S_0^2 \\ 0 \end{bmatrix}^T \begin{bmatrix} S_0^2 & 0 \\ 0 & 2S_0^4 \end{bmatrix} \begin{bmatrix} \sqrt{n}\overline{X}/S_0^2 \\ 0 \end{bmatrix} = \frac{n\left(\overline{X}\right)^2}{S_0^2}. \tag{4}$$

(iii) Finally, the Wald Statistic in this multiparameter setting is, from notes

$$W_n = \sqrt{n}\left(\widetilde{\theta}_{n1} - \theta_{10}\right)^T \left[\hat{I}_n^{(11.2)}\left(\widetilde{\theta}_n\right)\right] \sqrt{n}\left(\widetilde{\theta}_{n1} - \theta_{10}\right).$$

Here, as

$$\widetilde{\theta}_{n1} = \overline{X} \qquad \theta_{10} = 0 \qquad \hat{I}_n^{(11.2)}\left(\widetilde{\theta}_n\right) = I_{11} - I_{12}\left[I_{22}\right]^{-1} I_{21} = \frac{1}{\widehat{\sigma}^2} - 0$$

(as $I_{12} = I_{21} = 0$, where now $\widehat{\sigma}$ is estimated **under $\mathbf{H}_1$**, so that $\widehat{\sigma}^2 = S^2$), we thus have

$$W_n = \sqrt{n}\left(\overline{X}\right)^T \left[\frac{1}{S^2}\right] \sqrt{n}\left(\overline{X}\right) = \frac{n\left(\overline{X}\right)^2}{S^2} \tag{5}$$

Under H_0, all three statistics (3), (4) and 5) have a χ_1^2 distribution asymptotically.

(c) (i) Under H_0, the μ and σ^2 are completely specified, whereas under H_1, the MLEs of μ and σ^2 are, as in (b),

$$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i \qquad S^2 = \frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}\right)^2.$$

Thus

$$\begin{aligned} T_n &= \frac{L\left(\overline{X}, S^2\right)}{L\left(0, \sigma_0^2\right)} = \frac{\left(\dfrac{1}{2\pi S^2}\right)^{n/2} \exp\left\{-\dfrac{1}{2S^2}\sum\limits_{i=1}^n \left(X_i - \overline{X}\right)^2\right\}}{\left(\dfrac{1}{2\pi\sigma_0^2}\right)^{n/2} \exp\left\{-\dfrac{1}{2\sigma_0^2}\sum\limits_{i=1}^n X_i{}^2\right\}} \\ &= \left(\frac{S^2}{2\pi\sigma_0^2}\right)^{-n/2} \exp\left\{\frac{1}{2\sigma_0^2}\sum_{i=1}^n X_i{}^2 - \frac{n}{2}\right\} \end{aligned}$$

and thus

$$\begin{aligned} 2\log T_n &= -n\log\left(\frac{S^2}{\sigma_0^2}\right) + \frac{1}{\sigma_0^2}\sum_{i=1}^n X_i{}^2 - n \\ &= -n\log\left(\frac{S^2}{\sigma_0^2}\right) + \frac{1}{\sigma_0^2}\left(nS^2 + n\left(\overline{X}\right)^2\right) - n \\ &= n\left(\frac{S^2}{\sigma_0^2} - 1 - \log\left(\frac{S^2}{\sigma_0^2}\right)\right) + \frac{n\left(\overline{X}\right)^2}{\sigma_0^2} \end{aligned} \tag{6}$$

(ii) For the **Rao** Statistic, under H_0

$$\left[\hat{I}_n\left(\widetilde{\theta}_n^{(0)}\right)\right]^{-1} = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & 2\sigma_0^4 \end{bmatrix}$$

and, from (1) and (2),

$$Z_{n0} = \frac{1}{\sqrt{n}} l_n'\left(\widetilde{\theta}_n^{(0)}\right) = \frac{1}{\sqrt{n}} \begin{bmatrix} \dfrac{1}{\sigma_0^2}\sum_{i=1}^{n} X_i \\ -\dfrac{n}{2\sigma_0^2} + \dfrac{1}{2\sigma_0^4}\sum_{i=1}^{n} X_i^2 \end{bmatrix} = \frac{1}{\sigma_0^2} \begin{bmatrix} \sqrt{n}\overline{X} \\ \dfrac{1}{2\sigma_0^2}\sum_{i=1}^{n} X_i^2 - \dfrac{n}{2} \end{bmatrix}$$

and thus

$$\begin{aligned} Z_{n0} &= \frac{1}{\sigma_0^4} \begin{bmatrix} \sqrt{n}\overline{X} \\ \dfrac{1}{2\sigma_0^2}\sum_{i=1}^{n} X_i^2 - \dfrac{n}{2} \end{bmatrix}^T \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & 2\sigma_0^4 \end{bmatrix} \begin{bmatrix} \sqrt{n}\overline{X} \\ \dfrac{1}{2\sigma_0^2}\sum_{i=1}^{n} X_i^2 - \dfrac{n}{2} \end{bmatrix} \\ &= \frac{1}{\sigma_0^4}\left(n\sigma_0^2\left(\overline{X}\right)^2 + \frac{\sigma_0^4}{2n}\left(\frac{1}{\sigma_0^2}\sum_{i=1}^{n} X_i^2 - n\right)^2\right) = \frac{n\left(\overline{X}\right)^2}{\sigma_0^2} + \frac{n}{2}\left(\frac{1}{n\sigma_0^2}\sum_{i=1}^{n} X_i^2 - 1\right)^2 \qquad (7) \end{aligned}$$

(iii) The Wald Statistic is

$$\begin{aligned} W_n &= \sqrt{n}\left(\widetilde{\theta}_n - \theta_0\right)^T \left[\hat{I}_n\left(\widetilde{\theta}_n\right)\right] \sqrt{n}\left(\widetilde{\theta}_n - \theta_0\right) \\ &= \begin{bmatrix} \sqrt{n}\left(\overline{X} - 0\right) \\ \sqrt{n}\left(S^2 - \sigma_0^2\right) \end{bmatrix}^T \begin{bmatrix} 1/S^2 & 0 \\ 0 & 1/\left(2S^4\right) \end{bmatrix} \begin{bmatrix} \sqrt{n}\left(\overline{X} - 0\right) \\ \sqrt{n}\left(S^2 - \sigma_0^2\right) \end{bmatrix} \\ &= \frac{n\left(\overline{X}\right)^2}{S^2} + \frac{n\left(S^2 - \sigma_0^2\right)^2}{2S^4} \qquad (8) \end{aligned}$$

and again, under H_0, all three statistics (6), (7) and (8) have a χ_1^2 distribution asymptotically.

2. $Y_i \sim Poisson\left(N\theta\right)$

$$f_{Y_i|\theta}\left(y;\theta\right) = \frac{e^{-N\theta}\left(N\theta\right)^y}{y!} \qquad y = 0, 1, 2, ...$$

and a $Gamma\left(\alpha, \beta\right)$ prior

$$p_\theta\left(\theta\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)}\theta^{\alpha-1}e^{-\beta\theta} \qquad \theta > 0$$

combine to give a posterior

$$\begin{aligned} p_{\theta|Y}\left(\theta|y\right) &\propto f_{Y_i|\theta}\left(y;\theta\right)p_\theta\left(\theta\right) \propto e^{-N\theta}\left(N\theta\right)^y \times \theta^{\alpha-1}e^{-\beta\theta} \\ &\propto \theta^{y+\alpha-1}e^{-(N+\beta)\theta} \end{aligned}$$

and thus

$$\theta|Y = y \equiv Gamma\,(y+\alpha, N+\beta)$$

and hence

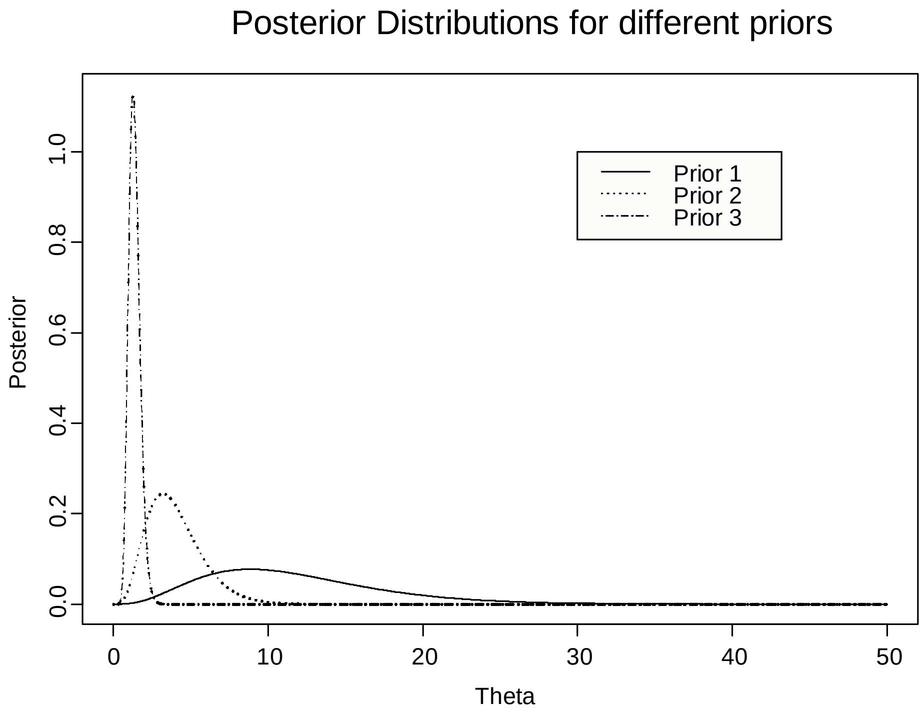
$$POSTERIOR\,MODE \quad : \quad \widehat{\theta}_{MODE} = \frac{y+\alpha-1}{N+\beta}$$

$$POSTERIOR\,MEAN \quad : \quad \widehat{\theta}_{MEAN} = \frac{y+\alpha}{N+\beta}$$

$$POSTERIOR\,VARIANCE \quad : \quad \frac{y+\alpha}{(N+\beta)^2}$$

If $y = 4$, $N = 0.25$

	$\widehat{\theta}_{MODE}$	$\widehat{\theta}_{MEAN}$	VAR
PRIOR 1	8.86	11.71	33.47
PRIOR 2	3.20	4.00	3.20
PRIOR 3	1.27	1.37	0.13



Clearly here the prior is very influential; with a fairly non-informative prior (PRIOR 1), the posterior is also fairly disperse, whereas with an informative prior (PRIOR 3) the posterior has low dispersion.

Note for comparison that the ML estimate of θ is $t = y/N = 16$, with associated standard error

$$\sqrt{\frac{Var_{f_{Y|\theta}}[Y]}{N^2}} = \sqrt{\frac{\theta}{N^2}} \qquad \therefore \qquad e.s.e\,(T) = \sqrt{\frac{\widehat{\theta}}{N^2}} = \sqrt{\frac{y}{N^3}} = 16$$

which is the equivalent of a Bayesian result with $\alpha = \beta = 0$.

3. As given in lectures

- **Likelihood: BINOMIAL**

$$f_{X|\theta}(x;\theta) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x}$$

- **Prior: BETA**

$$p_{\theta}(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

- **Posterior :**

$$p_{\theta|X}(\theta|x) \propto f_{X|\theta}(x;\theta)\, p_{\theta}(\theta) \propto \theta^{x}(1-\theta)^{n-x} \times \theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$$

so that the posterior is also **BETA**

$$p_{\theta|X}(\theta|x) \equiv Beta(x+\alpha, n-x+\beta)$$

Hence, here as the data within the rows are independently Binomially distributed with $\theta = \gamma_1$ and $\theta = \gamma_0$ respectively.

$$\begin{aligned} \gamma_1|N_{11} &= n_{11} \sim Beta(n_{11}+\alpha, n_{12}+\beta) \\ \gamma_0|N_{21} &= n_{21} \sim Beta(n_{21}+\alpha, n_{22}+\beta) \end{aligned} \tag{9}$$

with $\alpha = \beta = 1$. From standard results, for a $Beta(\alpha,\beta)$ random variable

$$MEAN : \frac{\alpha}{\alpha+\beta} \qquad VARIANCE : \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Thus the posterior means/variances are

$$\gamma_1\ MEAN : \frac{n_{11}+1}{n_{1.}+2} = \frac{96+1}{96+104+2} = 0.480$$

$$VAR : \frac{(n_{11}+1)(n_{12}+1)}{(n_{1.}+2)^2(n_{1.}+3)} = \frac{(96+1)(104+1)}{(96+104+2)^2(96+104+3)} = 1.23 \times 10^{-3}$$

$$\gamma_0\ MEAN : \frac{n_{21}+1}{n_{2.}+2} = \frac{109+1}{109+666+2} = 0.142$$

$$VAR : \frac{(n_{21}+1)(n_{22}+1)}{(n_{2.}+2)^2(n_{2.}+3)} = \frac{(109+1)(666+1)}{(109+666+2)^2(109+666+3)} = 1.56 \times 10^{-4}$$

whereas the "pure likelihood" approach has $\alpha = \beta = 0$, so that

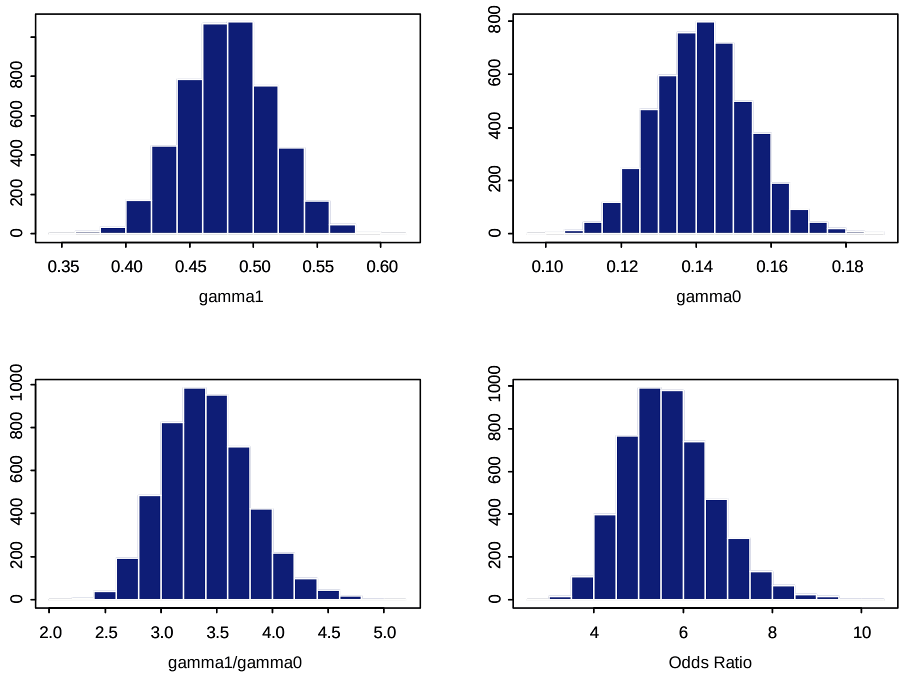
$$\gamma_1\ MEAN : \frac{96}{96+104} = 0.480 \qquad VAR := \frac{(96)(104)}{(96+104)^2(96+104+1)} = 1.24 \times 10^{-3}$$

$$\gamma_0\ MEAN : \frac{109}{109+666} = 0.141 \qquad VAR : \frac{n_{21}n_{22}}{n_{2.}^2(n_{2.}+1)} = \frac{(109)(666)}{(109+666)^2(109+666+1)} = 1.55 \times 10^{-4}$$

The results are fairly similar as we would expect when the sample sizes are so large. The prior with $\alpha = \beta = 1$ is perhaps not sensible, as it is uniform, and thus places too much probability on the range, say, $(0.5, 1)$. A more sensible prior might be $\alpha = 4$, $\beta = 10$.

Using simulation-based inference, it is possible to derive posterior samples (and hence posterior summaries) for derived parameters such as the odds ratio. Using the SPLUS function `rbeta` we can produce samples from each distribution in . We then combine the sampled values using the odds ratio formula, and summarize the samples for the derived parameter. If $\alpha = \beta = 1$, this proceeds using the following SPLUS code to produce samples of size 5000 from the posterior distributions of γ_1, γ_0, γ_1/γ_0 and $\psi = (\gamma_1/(1-\gamma_1))\,/\,(\gamma_0/(1-\gamma_0))$

```
n11<-96
n12<-104
n21<-109
n22<-666
g1<-rbeta(5000,n11+1,n12+1)
g0<-rbeta(5000,n21+1,n22+1)
par(mfrow=c(2,2))
hist(g1,xlab=''gamma1'')
axes()
hist(g0,xlab=''gamma0'')
axes()
hist(g1/g0,xlab=''gamma1/gamma0'')
axes()
hist((g1/(1-g1))/(g0/(1-g0)),xlab=''Odds Ratio'')
axes()
par(mfrow=c(1,1))
```



4. The likelihood and prior in the normal linear model $Y|X,\beta,\sigma^2 \sim N_n\left(X\beta, \sigma^2 1_n\right)$ are as follows:

- **Likelihood:**

$$f_{Y|X,\beta,\sigma^2}\left(y;X,\beta,\sigma^2\right) = \left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\sigma^2}\right)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\left(y-X\beta\right)^T\left(y-X\beta\right)\right\} \tag{10}$$

- **Prior:**

$$p_{\beta|\sigma^2}\left(\beta|\sigma^2\right) = \left(\frac{1}{2\pi}\right)^{(K+1)/2}\frac{1}{|\Sigma|^{1/2}}\left(\frac{1}{\sigma^2}\right)^{(K+1)/2}\exp\left\{-\frac{1}{2\sigma^2}\left(\beta-\theta\right)^T\Sigma^{-1}\left(\beta-\theta\right)\right\} \tag{11}$$

$$p_{\sigma^2}\left(\sigma^2\right) = \frac{\beta_\sigma^{\alpha_\sigma}}{\Gamma\left(\alpha_\sigma\right)}\left(\frac{1}{\sigma^2}\right)^{\alpha_\sigma+1}\exp\left\{-\frac{\beta_\sigma}{\sigma^2}\right\} \tag{12}$$

To compute the posterior, we combine these three distributions multiplicatively. Up to proportionality, therefore, we have the posterior as

$$\left(\frac{1}{\sigma^2}\right)^{(n+K+1)/2+\alpha_\sigma+1}\exp\left\{-\frac{1}{2\sigma^2}\left[\left(y-X\beta\right)^T\left(y-X\beta\right)+\left(\beta-\theta\right)^T\Sigma^{-1}\left(\beta-\theta\right)+2\beta_\sigma\right]\right\}$$

We combine terms in the quadratic term form for β: first, completing the square

$$\left(y-X\beta\right)^T\left(y-X\beta\right)+\left(\beta-\theta\right)^T\Sigma^{-1}\left(\beta-\theta\right) = \left(\beta-v\right)^T V\left(\beta-v\right)+c$$

where we must have (by multiplying out and equating terms in $\beta^T A\beta$, β and the constant)

$$\begin{aligned} V &= X^TX+\Sigma^{-1} \\ v &= \left(X^TX+\Sigma^{-1}\right)^{-1}\left(X^Ty+\Sigma^{-1}\mu\right) \\ c &= -v^TVv+\left[y^Ty+\theta^T\Sigma^{-1}\theta\right] \end{aligned}$$

Thus we have the posterior as proportional to

$$\left(\frac{1}{\sigma^2}\right)^{(n+K+1)/2+\alpha_\sigma+1}\exp\left\{-\frac{1}{2\sigma^2}\left[\left(\beta-v\right)^T V\left(\beta-v\right)+c+2\beta_\sigma\right]\right\} \tag{13}$$

Integrating out β, we get the marginal posterior for σ^2; this integral is tractable, as (13) is proportional to the Multivariate normal distribution

$$N_{K+1}\left(v,\sigma^2V\right). \tag{14}$$

On integrating out β, we obtain a constant of integration

$$\left(\sigma^2\right)^{(K+1)/2}|V|^{1/2}$$

and thus the marginal posterior for σ^2 is of the form

$$\left(\frac{1}{\sigma^2}\right)^{(n+K+1)/2+\alpha_\sigma+1}\exp\left\{-\frac{1}{2\sigma^2}\left[c+2\beta_\sigma\right]\right\}\times\left(\sigma^2\right)^{(K+1)/2} = \left(\frac{1}{\sigma^2}\right)^{n/2+\alpha_\sigma+1}\exp\left\{-\frac{1}{2\sigma^2}\left[c+2\beta_\sigma\right]\right\}$$

which is proportional to the Inverse Gamma pdf

$$IGamma\left(\frac{n}{2}+\alpha_\sigma,\frac{c}{2}+\beta_\sigma\right). \tag{15}$$

Hence, from (15) gives the marginal for σ^2, and (14) gives the conditional posterior for β given σ^2 (and X, y, and the prior). In the notation of the question

$$\theta^* = v \qquad \Sigma^* = V^{-1}$$

$$\alpha_\sigma^* = \frac{n}{2}+\alpha_\sigma \qquad \beta_\sigma^* = \frac{c}{2}+\beta_\sigma$$

5. The SPLUS code/output attached summarizes the analysis and results: key points are

- both β_0 and β_1 are significantly different from zero; using a Wald-type test

$$H_0 \ : \ \beta_0 = 0 \qquad \frac{\widehat{\beta}_0}{e.s.e\left(\widehat{\beta}_0\right)} = 8.325 \qquad p\text{-value is } 0.0000$$

$$H_0 \ : \ \beta_1 = 0 \qquad \frac{\widehat{\beta}_1}{e.s.e\left(\widehat{\beta}_1\right)} = 5.1151 \qquad p\text{-value is } 0.0005$$

- The ANOVA table for the test of the full regression model against the null model is

Source	D.F.	Sum of squares	Mean square	F	p-value
x	1	122.7772	122.7772	26.1643	4.54×10^{-4}
Residual	10	46.9254	4.6925		
Total	11	168.4697			

  so the regression model with β_1 included fits better than the null model

- The estimates and standard errors of the parameters are

$$\widehat{\beta}_0 \ : \ 9.5158 \ (1.1430)$$

$$\widehat{\beta}_1 \ : \ 0.5124 \ (0.1002)$$

- The unbiased estimate of σ^2 is

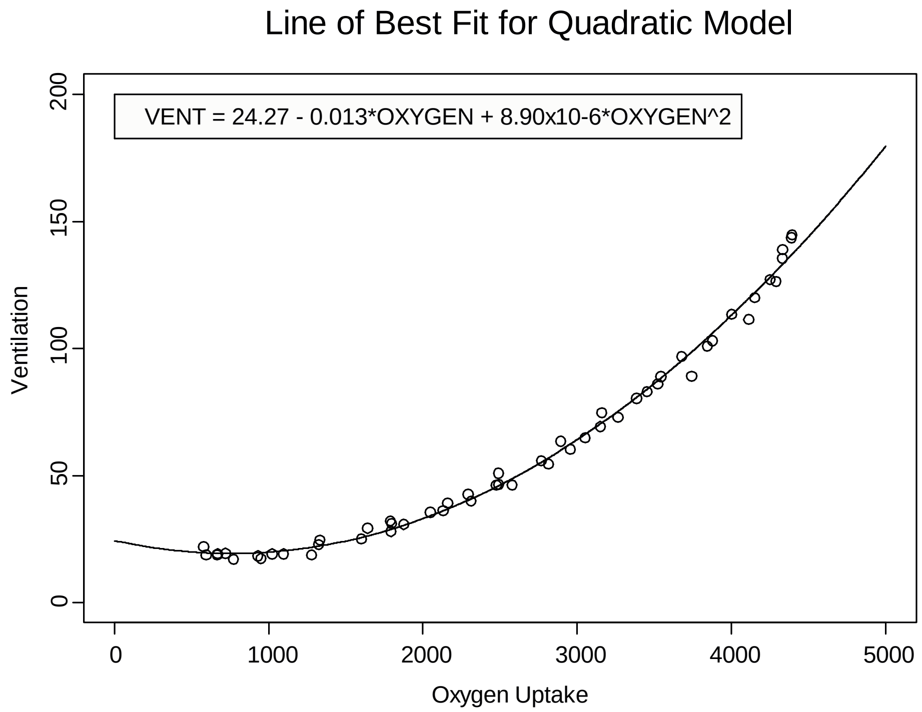
$$\frac{\text{Residual Sum of Squares}}{\text{Residual Degrees of Freedon}} = \frac{46.9254}{10} = 4.6925 = (2.166)^2$$

6. From a plot of the data, it is evident that a quadratic model should be fitted, that is

$$\text{VENTILATION} = \beta_0 + \beta_1 \times \text{OXYGEN} + \beta_2 \times \text{OXYGEN}^2$$

This can be readily fitted (within the linear model framework, as the model is linear in the elements of the parameter vector). The SPLUS sheet attached gives the necessary code. In particular, the parameter estimates/e.s.e, t-value (the test statistic in a Wald-type test) and p-value are as follows

	Estimate	Std. Error	t-value	p-value
β_0	2.427×10^{1}	1.94	12.51	0.000
β_1	-1.344×10^{-2}	1.76×10^{-3}	−7.63	0.000
β_2	8.902×10^{-6}	3.44×10^{-3}	25.85	0.000


Line of Best Fit for Quadratic Model
VENT = 24.27 - 0.013*OXYGEN + 8.90x10-6*OXYGEN^2
Ventilation
0
50
100
150
200
0
1000
2000
3000
4000
5000
Oxygen Uptake