M2S1 PROBABILITY AND STATISTICS

SHORT REVISION QUESTIONS

Chapter 1. Definitions, Terminology, Notation

1. What is the **sample space** of an experiment ?

2. In probability theory, what is an **event** ?

3. What are the three basic set operations that are used in probability theory ?

4. If events E, F are **mutually exclusive**, what can you say about $E \cap F$?

5. Define a **partition** of an event F.

6. Why do **partitions** of sample spaces attract particular attention in probability calculations ?

7. How can you construct a **disjoint** union from a general union of events ? That is, given a collection of events, $A_1, ..., A_k$, construct a collection $E_1, ..., E_k$ such that $E_i \cap E_j = \emptyset$ for all $i \neq j$ and

$$\bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} A_k$$

8. What is meant by saying that $E_1, E_2, ...$ is a **monotone non-increasing/non-decreasing** sequence of events ?

9. Define the limit event, E, of a monotone non-increasing sequence of events. Similarly, define the limit event of a monotone non-decreasing sequence. What are the interpretations of these limit events in terms of sample outcomes ?

10. If E is the limit event of a monotone sequence of events, to what does the event "E occurs" correspond if the sequence is non-increasing/non-decreasing

11. For a **general** infinite sequence of events $E_1, E_2, ...$, show how to **construct** a unique monotone non-increasing sequence and a unique monotone non-decreasing sequence.

12. What are the three **axioms of probability** ?

13. Define the **conditional probability** of E given F.

14. If events E, F are **independent**, what can you say about $\mathrm{P}(E \cap F)$?

15. If events E, F are **not** independent, what can you say about $\mathrm{P}(E \cap F)$?

16. What is the **Theorem of Total Probability** ?

17. What is **Bayes Theorem** ?

Chapter 2. Probability Distributions and Random Variables

18. What is a **random variable** ?

19. What is the difference between a **discrete** and a **continuous** random variable ?

20. For a **discrete** random variable X, define the **probability mass function**, f_X, of X. Which two conditions must f_X satisfy ?

21. For a discrete random variable X, define the **cumulative distribution function**, or **cdf**, F_X, of X. What properties does F_X exhibit ?

22. For a discrete random variable X, how are f_X and F_X related ?

23. For a **continuous** random variable X, define the **cumulative distribution function**, or **cdf**, F_X, of X. What properties does F_X exhibit ?

24. For a **continuous** random variable X, define the **probability density function**, or **pdf**, f_X, of X. Which two conditions must f_X satisfy ?

25. For a continuous discrete random variable X, how are f_X and F_X related ?

26. Define the **expectation** of a function $g(X)$ of random variable X, $\mathrm{E}_{f_X}[X]$. Hence define the **variance** of a random variable X, $\mathrm{Var}_{f_X}[X]$.

27. What are the expectation and variance of $2(X+2)$ in terms of the expectation and variance of X ?

28. What is the k**th moment** of a probability distribution ?

29. What is the k**th central moment** of a probability distribution ?

30. If X is a random variable with cdf F_X, and Y is a random variable defined by $Y = g(X)$ for some function g, how is the distribution of Y calculated ?

31. If discrete random variables X and Y are related by $Y = g(X)$, and g is 1-1, how is mass function f_Y calculated from f_X ?

32. If continuous random variables X and Y are related by $Y = g(X)$, and g is 1-1, how is density function f_Y calculated from f_X ?

33. What is the **Jacobian**, $J(y)$, of a transformation g, a 1-1 transformation from $x \in \mathbb{X}$ to $y \in \mathbb{Y}$?

34. For discrete random variables $X_1, ..., X_k$, define the **joint mass function**, $f_{X_1,...,X_k}$, and the **joint cumulative distribution function**, $F_{X_1,...,X_k}$, and explain how to calculate $f_{X_1,...,X_k}$ from $F_{X_1,...,X_k}$ and *vice versa.*

35. For continuous random variables $X_1, ..., X_k$, define the **joint cumulative distribution function**, $F_{X_1,...,X_k}$, and the **joint density function**, $f_{X_1,...,X_k}$, and explain how to calculate $f_{X_1,...,X_k}$ from $F_{X_1,...,X_k}$ and *vice versa.*

36. For discrete random variables $X_1, ..., X_k$ with joint mass function $f_{X_1,...,X_k}$, how is the **marginal mass function** of X_i, f_{X_i}, calculated ?

37. For continuous random variables $X_1, ..., X_k$ with joint density function $f_{X_1,...,X_k}$, how is the **marginal density function** of X_i, f_{X_i}, calculated ?

38. For random variables $X_1, ..., X_k$ with joint mass/density function $f_{X_1,...,X_k}$, define the **conditional mass/density function** of X_i given the other variables $X_1, ..., X_{i-1}, X_{i+1}, ..., X_k$.

39. What are the necessary and sufficient conditions that must hold if random variables $X_1, ..., X_k$ are **independent** ?

40. What is the **Chain Rule** for random variables ?

41. What is the **Law of Iterated Expectation** ?

42. For a vector of k continuous random variables $\mathbf{X} = (X_1, ..., X_k)$ with joint density function $f_{X_1,...,X_k}$, consider the vector of continuous random variables $\mathbf{Y} = (\mathbf{Y_1}, ..., \mathbf{Y_k})$, defined by $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ for some 1-1 function $\mathbf{g}$. How is the joint density function of $\mathbf{Y}$, $f_{Y_1,...,Y_k}$, calculated ?

43. What are the **covariance** and **correlation** of two random variables ? What is the connection between the covariance and the **dependence** of two random variables ?

44. Why is the correlation a more informative measure of the association between two variables than the covariance ?

45. If random variable $X = X_1 + ... + X_k$, how can the expectation and variance of X be calculated in terms of the expectations, variances, and covariances of $X_1, ..., X_k$?

46. What is the **convolution formula**, and what is it used for ?

47. For random variables $X_1, ..., X_k$, define the **order statistics** $Y_1, ..., Y_k$ derived from $X_1, ..., X_k$. If random variables $X_1, ..., X_k$ are independent and identically distributed with mass/density function f_X, and $Y_1, ..., Y_k$ are the corresponding order statistics, write down

(i) the **joint** mass/density function of $Y_1, ..., Y_k$,
(ii) the **marginal** mass/density function for Y_i,
(iii) the **joint** mass/density function for Y_i and Y_j $(i < j)$.

Derive the distribution of the **minimum** and **maximum** order statistics, Y_1 and Y_k corresponding to an independent and identically distributed collection of random variables $X_1, ..., X_k$.

48. For random variable X with mass/density function f_X, define the **moment generating function**, or mgf, of X, M_X, and the **probability generating function**, or pgf, of X, G_X. How is the mgf M_X used to calculate the moments of f_X ?

49. How is the mgf M_Y related to the mgf M_X if X and Y are related by $Y = aX + b$?

50. If $X_1, ..., X_k$ are independent random variables, with mgfs M_{X_i} for $i = 1, ..., k$ respectively, and $Y = X_1 + ...X_k$, what is the mgf of Y, M_Y ? What is the form of M_Y if $X_1, ..., X_k$ are identically distributed ?

Chapter 3. Discrete probability distributions

51. What is the form of the mgf for the **discrete uniform distribution** ?

52. What is the form of the mgf for the **Bernoulli distribution** ?

53. What is the form of the mgf for the **binomial distribution** ?

54. How are the Bernoulli and binomial distributions related ?

55. What is the form of the mgf for the **Poisson distribution** ?

56. How are the binomial and Poisson distributions related ?

57. What is the distribution of $Y = X_1 + X_2$, where $X_1 \sim Poisson(\lambda_1), X_2 \sim Poisson(\lambda_2)$?

58. What four assumptions underlie the Poisson process ?

59. How are the Poisson process and the Poisson distribution related ?

60. What is the form of the mgf for the **geometric distribution** ?

61. How is the geometric distribution unique among discrete distributions ?

62. How are the Bernoulli and the geometric distributions related ?

63. What is the form of the mgf for the **negative binomial distribution** ?

64. How are the binomial and the negative binomial distributions related ?

65. How are the Bernoulli and the negative binomial distributions related ?

66. How are the geometric and the negative binomial distributions related ?

67. What is the alternative representation of the negative binomial distribution, and to what experimental situation does it correspond ?

68. How are the Poisson and the negative binomial distributions related ?

69. How are the **hypergeometric** and the binomial distributions related ?

Chapter 4. Continuous probability distributions

70. What is the form of the mgf for the **continuous uniform distribution** ?

71. What is the form of the mgf for the **exponential distribution** ?

72. How are the Poisson process and the exponential distribution related ?

73. What is the **lack of memory property** ?

74. How are the continuous uniform and the exponential distributions related ?

75. What is the form of the mgf for the **gamma distribution** ?

76. What is the distribution of $Y = X_1 + X_2$, where $X_1 \sim Ga(\alpha_1, \beta), X_2 \sim Ga(\alpha_2, \beta)$?

77. How are the gamma and **chi-square** distributions related ?

78. If independent random variables $X_1, ..., X_n$ have chi-square distributions with degrees of freedom $m_1, ..., m_n$ respectively, what is the distribution of $Y = X_1 + ... + X_n$?

79. How are the Poisson, gamma, and negative binomial distributions related ?

80. How are the gamma and **beta** distributions related ?

81. How are the beta and the continuous uniform distributions related ?

82. What is the form of the mgf for the **standard normal distribution** ?

83. What is the form of the mgf for the $Normal(\mu, sigma^2)$ distrubution ?

84. What is the distribution of $Y = X_1 + X_2$, where $X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2)$?

85. State, and explain the implication of, the **Central Limit Theorem** ?

86. Explain the method of proof used to prove the Central Limit Theorem in general, and for specific distributions of the collection $X_1, ...X_n$.

87. Show how to use the Central Limit Theorem to construct approximations to standard distributions, such as Binomial, Poisson, Gamma and Normal.

88. What is the distribution of $Y = X^2$, where $X \sim N(0, 1)$?

89. How are the normal and **Cauchy** distributions related ?

90. How are the Cauchy and **Student-t** distributions related ?

91. How are the normal and Student-t distributions related ?

92. What is the distribution of $Y = aX + b$ where $X \sim N(0, 1)$?

Chapter 5. Probability results and Limit Theorems

93. What is the **Chebychev Inequality** ?

94. What is **Jensen's Inequality** ?

95. What is meant by "the sequence of random variables **converges in distribution** to random variable X" ?

96. What is the **limiting distribution** of a sequence of random variables ?

97. What is meant by "the sequence of random variables **converges stochastically** to c" ?

98. What is meant by "the sequence of random variables **converges in probability** to c" ?

99. What is the **Weak Law of Large Numbers** ?

100. What is meant by "the sequence of random variables **converges in probability** to random variable X" ?