THE CENTRAL LIMIT THEOREM

THEOREM

Suppose $X_1, ..., X_n$ are i.i.d. random variables with mgf M_X, with

$$\mathrm{E}_{f_X}[X_i] = \mu \qquad \mathrm{Var}_{f_X}[X_i] = \sigma^2$$

Let the random variable Z_n be defined by

$$Z_n = \frac{\displaystyle\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$$

and let Z_n have mgf M_{Z_n}. Then, as $n \longrightarrow \infty$,

$$M_{Z_n}(t) \longrightarrow \exp\left\{\frac{t^2}{2}\right\}$$

irrespective of the form of M_X.

PROOF:

First, let $Y_i = (X_i - \mu)/\sigma$ for $i = 1, ..., n$. Then $Y_1, ..., Y_n$ are i.i.d. with mgf M_Y say, and by the elementary properties of expectation, $\mathrm{E}_{f_Y}[Y_i] = 0, \mathrm{Var}_{f_Y}[Y_i] = 1$ for each i. Using the power series expansion result for mgfs, we have that

$$\begin{aligned} M_Y(t) &= 1 + tE_{f_Y}[Y] + \frac{t^2}{2!}E_{f_Y}[Y^2] + \frac{t^3}{3!}E_{f_Y}[Y^3] + ... \\ &= 1 + \frac{t^2}{2!} + \frac{t^3}{3!}E_{f_Y}[Y^3] + ... \end{aligned}$$

Now, the random variable Z_n can be rewritten

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$$

and thus, again by a standard mgf result, as $Y_1, ..., Y_n$ are independent, we have that

$$M_{Z_n}(t) = \prod_{i=1}^{n} \left\{M_Y(t/\sqrt{n})\right\} = \left\{1 + \frac{t^2}{2n} + \frac{t^3}{6n^{3/2}}E_{f_Y}[Y^3] + ...\right\}^n$$

Taking logs, and using the expansion $\ln(1+s) = s - s^2/2 + s^3/3 - ...$ we have that

$$\ln M_{Z_n}(t) = n\left[\left(\frac{t^2}{2n} + \frac{t^3}{6n^{3/2}}E_{f_Y}[Y^3] + ...\right) - \frac{1}{2}\left(\frac{t^2}{2n} + \frac{t^3}{6n^{3/2}}E_{f_Y}[Y^3] + ...\right)^2 + ...\right]$$

Thus, as $n \longrightarrow \infty$, only the very first term is non-zero, and

$$\ln M_{Z_n}(t) \longrightarrow \frac{t^2}{2} \quad \text{so that} \quad M_{Z_n}(t) \longrightarrow \exp\left\{\frac{t^2}{2}\right\}$$