M2S1 : EXERCISE SHEET 8 : SOLUTIONS

1. As $X_1, ..., X_n \sim Poisson(\lambda)$, and given that $T_1 = \bar{X}$, then using elementary properties of expectations, we have

$$E_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n} E_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\lambda = \lambda$$

so that T_1 is an *unbiased* estimator of λ. Furthermore

$$\begin{aligned} T_2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\left(\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} X_i\right)^2 \end{aligned}$$

From properties of expectations, variances, and the Poisson distribution,

$$E_{f_X}\left[X^2\right] = Var_{f_X}[X] + \left\{E_{f_X}[X]\right\}^2 = \lambda + \lambda^2 = \lambda(\lambda+1)$$

Now, from properties of independent Poisson random variables $Y_n = \sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$ so therefore, taking expectations in the above

$$\begin{aligned} E_{f_{T_2}}[T_2] &= \frac{1}{n-1}\sum_{i=1}^{n} E_{f_{X_i}}\left[X_i^2\right] - \frac{1}{n(n-1)}E_{f_{Y_n}}\left[Y_n^2\right] = \frac{1}{n-1}\sum_{i=1}^{n}\lambda(\lambda+1) - \frac{1}{n(n-1)}n\lambda(n\lambda+1) \\ &= \frac{n}{n-1}\lambda(\lambda+1) - \frac{1}{n-1}\lambda(n\lambda+1) = \frac{1}{n-1}\left[n\lambda(\lambda+1) - \lambda(n\lambda+1)\right] = \lambda \end{aligned}$$

2. From the theorem in lectures, and by properties of the Gamma distribution, we can write

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2_{n-1} \equiv Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right) \Longrightarrow V_n = \sum_{i=1}^{n} X_i$$

where $X_i \sim \chi^2_1 \equiv Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, so that, using the Gamma expectation and variance results

$$\mathrm{E}_{f_{X_i}}[X_i] = \frac{1/2}{1/2} = 1 = \mu \qquad \mathrm{Var}_{f_{X_i}}[X_i] = \frac{1/2}{(1/2)^2} = 2 = \sigma^2$$

say. Hence, by the Central Limit Theorem,

$$\frac{V_n - (n-1)\mu}{\sqrt{(n-1)\sigma^2}} = \frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0,1)$$

Hence, substituting in the definition for V_n,

$$\frac{\dfrac{(n-1)s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}\left(s_n^2 - \sigma^2\right)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1)$$

and finally, by a location/scale transformation to $Z_n = \sigma^2 + \frac{\sigma^2\sqrt{2}}{\sqrt{n-1}}Z$, we have

$$s_n^2 \xrightarrow{d} Z_n \sim N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

3. $X_1, ..., X_n \sim Gamma(\alpha, \beta)$ so that $\mathrm{E}_{f_{X_i}}[X_i] = \alpha/\beta$ and $\mathrm{Var}_{f_{X_i}}[X_i] = \alpha/\beta^2$ so that

$$E_{f_{X_i}}\left[X_i^2\right] = \mathrm{Var}_{f_{X_i}}[X_i] + \left\{\mathrm{E}_{f_{X_i}}[X_i]\right\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha+1)}{\beta^2}$$

Hence for the method of moments estimators $\hat{\alpha}_{MM}$ and $\hat{\beta}_{MM}$, need to solve the following:

FIRST MOMENT Solve $$\frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} = \frac{\alpha}{\beta}$$

SECOND MOMENT Solve $$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = (\overline{x})^2 + S^2 = \frac{\alpha(\alpha+1)}{\beta^2}$$

where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\overline{x})^2$

Elementary algebra gives

$$\hat{\alpha}_{MM} = \frac{(\overline{x})^2}{S^2} \qquad \hat{\beta}_{MM} = \frac{\overline{x}}{S^2}$$

5. (i) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

STEP 2 $$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0 \implies \hat{\theta}_{ML} = -n/\sum_{i=1}^{n}\log x_i$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n}\log x_i} \qquad \text{ESTIMATOR: } -\frac{n}{\sum_{i=1}^{n}\log X_i}$$

(ii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n}(\theta+1)\,x_i^{-(\theta+2)} = (\theta+1)^n\left(\prod_{i=1}^{n} x_i\right)^{-(\theta+2)}$$

STEP 2 $$\log L(\theta) = n\log(\theta+1) - (\theta+2)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta+1} - \sum_{i=1}^{n}\log x_i = 0 \implies \hat{\theta}_{ML} = n/\sum_{i=1}^{n}\log x_i - 1$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{(\theta+1)^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1 \qquad \text{ESTIMATOR: } = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1$$

(iii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n}\left(\prod_{i=1}^{n} x_i\right)\exp\left\{-\theta\sum_{i=1}^{n} x_i\right\}$$

STEP 2 $$\log L(\theta) = 2n\log\theta + \sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^{n} x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{2n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^{n} x_i} \qquad \text{ESTIMATOR:} = \frac{2n}{\sum\limits_{i=1}^{n} X_i}$$

(iv) Because of the constraint in the pdf that $x \leq \theta$

STEP 1 $$L(\theta) \quad \prod_{i=1}^{n} f_X(x_i;\theta) == \begin{cases} \prod\limits_{i=1}^{n} 2\theta^2 x_i^{-3} = 2^n\theta^{2n}\left(\prod\limits_{i=1}^{n} x_i^{-3}\right) & \theta \leq x_1, ..., x_n \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log L(\theta) = n\log 2 + 2n\log\theta - 3\sum_{i=1}^{n}\log x_i$$

At this point we note that the likelihood is monotonically increasing in θ, and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \leq x_1, ..., x_n$ is still satisfied, hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \min\{x_1, ..., x_n\} \qquad \text{ESTIMATOR: } \min\{X_1, ..., X_n\}$$

(v) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n}\left(\frac{\theta}{2}\right)\exp\{-\theta|x_i|\} = 2^{-n}\theta^n\exp\left\{-\theta\sum_{i=1}^{n}|x_i|\right\}$$

STEP 2 $$\log L(\theta) = -n\log 2 + n\log\theta - \theta\sum_{i=1}^{n}|x_i|$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta} - \sum_{i=1}^{n}|x_i| = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^{n}|x_i|}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^{n}|x_i|} \qquad \text{ESTIMATOR}: \frac{n}{\sum\limits_{i=1}^{n}|x_i|}$$

(vi) Because of the constraint in the pdf that $\theta_1 \leq x \leq \theta_2$

STEP 1 $$L(\theta_1,\theta_2) \quad \prod_{i=1}^{n} f_X(x_i;\theta) = \begin{cases} \prod_{i=1}^{n} \dfrac{1}{(\theta_2-\theta_1)} = \dfrac{1}{(\theta_2-\theta_1)^n} & \theta_1 \leq x_1,...,x_n \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log L(\theta_1,\theta_2) \quad = -n\log(\theta_2-\theta_1)$$

At this point we note that the likelihood is monotonically increasing in $\theta_1$and monotonically decreasing in θ_2, and hence the likelihood is maximized when θ_1 is as **large** as possible and when θ_2 is as **small** as possible, but so that the constraint $\theta_1 \leq x_1,...,x_n \leq \theta_2$ (and $\theta_2 \geq \theta_1$) is still satisfied, hence

ESTIMATE : $$\hat{\theta}_{1ML} = \min\{x_1,...,x_n\} \qquad \hat{\theta}_{2ML} = \max\{x_1,...,x_n\}$$

ESTIMATOR: $$\theta_1 \quad \min\{X_1,...,X_n\} \qquad \theta_2 \quad \max\{X_1,...,X_n\}$$

(vii) Noting the constraint in the pdf that $x \geq \theta_2$, we have

STEP 1 $$L(\theta_1,\theta_2) \quad = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta_1\theta_2^{\theta_1}x_i^{-(\theta_1+1)} = \theta_1^n\theta_2^{n\theta_1}\left(\prod_{i=1}^{n} x_i\right)^{-(\theta_1+1)}$$

$$0 \leq x_1,...,x_n \leq \theta_2$$

STEP 2 $$\log L(\theta_1,\theta_2) \quad = n\log\theta_1 + n\theta_1\log\theta_2 - (\theta_1+1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{\partial}{\partial\theta_1}\{\log L(\theta_1,\theta_2)\} \quad = \frac{n}{\theta_1} + n\log\theta_2 - \sum_{i=1}^{n}\log x_i = 0$$

$$\Longrightarrow \quad \hat{\theta}_{1ML} = \frac{n}{\sum_{i=1}^{n}\log x_i - n\log\hat{\theta}_{2ML}}$$

$$\frac{\partial}{\partial\theta_2}\{\log L(\theta_1,\theta_2)\} \quad = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when θ_2 is as **large** as possible, that is, when $\hat{\theta}_{2ML} = \min\{x_1,...,x_n\}$ .Hence

ESTIMATES $$\hat{\theta}_{1ML} = \frac{n}{\left[\sum_{i=1}^{n}\log x_i - n\log\{\min\{x_1,...,x_n\}\}\right]} \qquad \hat{\theta}_{2ML} = \min\{x_1,...,x_n\}$$

ESTIMATORS: $$\theta_1 \quad \frac{n}{\left[\sum_{i=1}^{n}\log x_i - n\log\{\min\{X_1,...,X_n\}\}\right]} \qquad \theta_2 \quad \min\{X_1,...,X_n\}$$

5. Follow the four step procedure that can be summarized as follows: for an observed random sample $x_1,...,x_n$ from a distribution represented by mass/density function $f_X(x;\theta)$

STEP 1 : Form the likelihood function $L(\theta)$

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

STEP 2 Take (natural) log to obtain $\log L(\theta)$

STEP 3: Find the value of θ at which $\log L(\theta)$ (and hence $L(\theta)$) is maximized within the parameter space Θ by differentiation

STEP 4: Check the **maximum** value has been found.

Formally, we define the **maximum likelihood estimate** of θ, $\hat{\theta}_{ML}$, as

$$\hat{\theta}_{ML} = \arg\max_{\theta} L(\theta)$$

Hence, for the $Poisson(\lambda)$ case

STEP 1 $$L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\left(\prod_{i=1}^{n} x_i!\right)}$$

STEP 2 $$\log L(\lambda) = -\sum_{i=1}^{n} \log x_i! - n\lambda + \left(\sum_{i=1}^{n} x_i\right)\log\lambda$$

STEP 3 $$\frac{d}{d\lambda}\{\log L(\lambda)\} = -n + \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

STEP 4 $$\frac{d^2}{d\lambda^2}\{\log L(\lambda)\} = -\left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda^2} < 0 \qquad \text{for all } \lambda$$

Therefore

ESTIMATE : $\hat{\lambda}_{ML} = \overline{x}$ ESTIMATOR: $\overline{X}$

and from question 1 on this sheet, we know that if $T_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ then $E_{f_{T_1}}[T_1] = \lambda$ and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \qquad \text{so that} \qquad \lambda = -\log\tau$$

and we can reformulate the likelihood in terms of τ, giving

$$\log L(\tau) = -\sum_{i=1}^{n} \log x_i! + n\log\tau + \left(\sum_{i=1}^{n} x_i\right)\log(-\log\tau)$$

and

$$\frac{d}{d\lambda}\{\log L(\tau)\} = \frac{\left(\sum_{i=1}^{n} x_i\right)}{-\tau\log\tau} + \frac{n}{\tau} = 0 \Longrightarrow \qquad \hat{\tau}_{ML} = \exp\left\{-\frac{1}{n}\sum_{i=1}^{n} x_i\right\} = \exp\{-\overline{x}\}$$

which can be shown to be the value that maximizes the likelihood, so that

$$\hat{\tau}_{ML}(\lambda) = \tau\left(\hat{\lambda}_{ML}\right)$$

6. Noting the constraint in the pdf that $x \geq \eta$, we have

STEP 1 $$L(\lambda, \eta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \lambda \exp\{-\lambda(x_i - \eta)\} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n}(x_i - \eta)\right\}$$

$\eta \leq x_1, ..., x_n$, zero otherwise

STEP 2 $$\log L(\lambda, \eta) = n \log \lambda - \lambda \sum_{i=1}^{n}(x_i - \eta) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i + n\lambda\eta$$

STEP 3 $$\frac{\partial}{\partial \lambda}\{\log L(\lambda, \eta)\} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - n\eta = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i - n\hat{\eta}}$$

$$\frac{\partial}{\partial \eta}\{\log L(\lambda, \eta)\} = n\lambda$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when η is as **large** as possible, that is, when $\hat{\eta} = \min\{x_1, ..., x_n\}$ .Hence

ESTIMATES

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i - n\min\{x_1, ..., x_n\}} = \frac{n}{\sum_{i=1}^{n}(x_i - \min\{x_1, ..., x_n\})}$$

$$\hat{\eta} = \min\{x_1, ..., x_n\}$$

ESTIMATORS:

$$\lambda = \frac{n}{\sum_{i=1}^{n}(X_i - \min\{X_1, ..., X_n\})}$$

$$\eta \quad \min\{X_1, ..., X_n\}$$

7. $X_1, ..., X_n \sim Exponential(1/\theta)$ so that $\mathrm{E}_{f_{X_i}}[X_i] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^{n} X_i \sim Gamma\left(n, \frac{1}{\theta}\right) \qquad \Longrightarrow \qquad E_{f_X}[X] = \frac{n}{\frac{1}{\theta}} = n\theta$$

so that if $T_1 = \overline{X} = \dfrac{1}{n}X$ then

$$E_{f_{T_1}}[T_1] = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta} \qquad y > 0$$

so that $Y_1 \sim Exponential\left(\dfrac{n}{\theta}\right)$. Hence if $T_2 = nY_1$ then

$$E_{f_{Y_1}}[Y_1] = \frac{\theta}{n} \qquad \therefore \qquad E_{f_{T_2}}[T_2] = n\frac{\theta}{n} = \theta$$

and hence T_2 is an unbiased estimator of θ.

$$f_X(x) = \frac{1}{2} \qquad \theta - 1 \leq x \leq \theta + 1 \qquad\qquad F_X(x) = \frac{x - (\theta - 1)}{2} = \frac{x - \theta + 1}{2} \qquad \theta - 1 \leq x \leq \theta + 1$$

$E_{f_{X_i}}[X_i] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \overline{X}$

$$E_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n} E_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$and $Y_n = \max\{X_1, ..., X_n\}$, then previous order statistics results give that

$$f_{Y_1}(y) = nf_X(y)\{1 - F_X(y)\}^{n-1} = n\frac{1}{2}\left\{1 - \frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1} \qquad \theta-1 \le y \le \theta+1$$

and

$$f_{Y_n}(y) = nf_X(y)\{F_X(y)\}^{n-1} = n\frac{1}{2}\left\{\frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} \qquad \theta-1 \le y \le \theta+1$$

For the expectations,

$$\begin{aligned}
E_{f_{Y_1}}[Y_1] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 ((1+\theta)-2t)\,t^{n-1}\,2\,dt \qquad \text{setting } t = (1+\theta-y)/2 \therefore y = (1+\theta)-2t \\
&= (1+\theta)\int_0^1 nt^{n-1}\,dt - 2n\int_0^1 t^n dt \\
&= (1+\theta) - \frac{2n}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
E_{f_{Y_n}}[Y_n] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 (2t-(1-\theta))\,t^{n-1}\,2\,dt \qquad \text{setting } t = (1-\theta+y)/2 \therefore y = 2t-(1-\theta) \\
&= 2\int_0^1 nt^n\,dt - (1-\theta)\int_0^1 nt^{n-1}dt \\
&= \frac{2n}{n+1} - (1-\theta)
\end{aligned}$$

so that if $M = (Y_1 + Y_n)/2$ then by properties of expectations

$$E_{f_M}[M] = \frac{1}{2}E_{f_{Y_1}}[Y_1] + \frac{1}{2}E_{f_{Y_n}}[Y_n] = \left[\frac{1}{2}(1+\theta) - \frac{n}{n+1}\right] + \left[\frac{n}{n+1} - \frac{1}{2}(1-\theta)\right] = \theta$$

and hence M is an unbiased estimator for θ

9. (i) For $\lambda > 0$

STEP 1 $L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n} \frac{\lambda^2}{\Gamma(2)} x_i \exp\{-\lambda x_i\} = \frac{\lambda^{2n}}{\{\Gamma(2)\}^n} \left(\prod_{i=1}^{n} x_i\right) \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\}$

STEP 2 $\log L(\lambda) = 2n\log\lambda - n\log\Gamma(2) + \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i$

STEP 3 $\frac{d}{d\theta}\{\log L(\lambda)\} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \quad \Longrightarrow \quad \hat{\lambda}_{ML} = \frac{2n}{\sum_{i=1}^{n} x_i}$

STEP 4 $\frac{d^2}{d\theta^2}\{\log L(\lambda)\} = -\frac{2n}{\lambda^2} < 0 \quad$ at $\lambda = \hat{\lambda}_{ML}$

Hence the estimator is $2n/\left(\sum_{i=1}^{n} X_i\right)$

(ii) The result from q. 4 can be generalized so that for any function $\tau = \tau(\lambda)$

$$\hat{\tau}_{ML}(\lambda) = \tau\left(\hat{\lambda}_{ML}\right)$$

which is referred to as the **invariance** property of maximum likelihood estimators. Thus we must have that the ML estimator of $\tau = 1/\lambda$ is

$$T = \frac{1}{2n}\sum_{i=1}^{n} X_i$$

Now, using mgfs, it is straightforward to show that

$$\sum_{i=1}^{n} X_i \sim Gamma(2n,\lambda) \qquad \therefore \qquad T = \frac{1}{2n}\sum_{i=1}^{n} X_i \sim Gamma(2n, 2n\lambda)$$

so that

$$\mathrm{E}_{f_T}[T] = \frac{2n}{2n\lambda} = \frac{1}{\lambda} \qquad \mathrm{Var}_{f_T}[T] = \frac{2n}{(2n\lambda)^2} = \frac{1}{2n\lambda^2} \qquad \mathrm{E}_{f_T}[T^2] = \frac{1}{2n\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2n+1}{2n\lambda^2}$$

(iii) Using the Central Limit Theorem we have that

$$\frac{\sum_{i=1}^{n} X_i - \frac{2n}{\lambda}}{\sqrt{\frac{2n}{\lambda^2}}} \xrightarrow{d} Z \sim N(0,1)$$

so that, dividing through by $2n$

$$\frac{T - \frac{1}{\lambda}}{\sqrt{\frac{1}{2n\lambda^2}}} \xrightarrow{d} Z \sim N(0,1)$$

and hence, approximately

$$T \sim N\left(\frac{1}{\lambda}, \frac{1}{2n\lambda^2}\right)$$

and as $n \longrightarrow \infty$ then variance tends to zero, and hence for $\varepsilon > 0$

$$\mathrm{P}\left[\left|T - \frac{1}{\lambda}\right| < \varepsilon\right] \longrightarrow 1 \therefore T \xrightarrow{p} \frac{1}{\lambda}$$