M2S1 : EXERCISES 3 : SOLUTIONS

1. Clearly F_X is continuous, and if $c = 1$,

$$\lim_{x \longrightarrow -\infty} F_X(x) = 0 \qquad \lim_{x \longrightarrow \infty} F_X(x) = 1$$

so F_X is a cdf.. To find the pdf, differentiate F_X;

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x} = \frac{d}{dt}\left\{\exp\left\{-e^{-\lambda t}\right\}\right\}_{t=x} = \lambda \exp\left\{-\lambda x - e^{-\lambda x}\right\} \qquad x \in \mathbb{R}.$$

If $f_X(x) = cg(x)$ is a pdf, then the corresponding cdf F_X is defined by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt \ = \begin{cases} \displaystyle\int_{-\infty}^{x} -\frac{ct}{(1+t^2)^2}\ dt \\ \displaystyle\int_{-\infty}^{0} -\frac{ct}{(1+t^2)^2}\ dt + \int_0^x \frac{ct}{(1+t^2)^2}\ dt \end{cases}$$

$$= \begin{cases} \left[\dfrac{c}{2}\dfrac{1}{1+t^2}\right]_{-\infty}^{x} & x \le 0 \\ \dfrac{c}{2} + \left[-\dfrac{c}{2}\dfrac{1}{1+t^2}\right]_0^x & x > 0 \end{cases} = \begin{cases} \dfrac{c}{2(1+x^2)} & x \le 0 \\ \dfrac{c(1+2x^2)}{2(1+x^2)} & x > 0 \end{cases}$$

and hence $c = 1$, as we must have $\lim_{x \longrightarrow \infty} F_X(x) = 1$

$\mathrm{E}_{f_X}[\ X\ ] = 0$ as f_X is symmetric about 0, and the expectation integral is *finite*. We know that

$$\int_{-\infty}^{\infty} x f_X(x)\ dx = \int_{-\infty}^{0} \frac{-x^2}{(1+x^2)^2}\ dx + \int_0^{\infty} \frac{x^2}{(1+x^2)^2}\ dx = 0$$

as the integrands in these integrals behave like $1/x^2$ as x becomes large, and hence the integrals are finite, and cancel as they are equal and opposite in sign.

2.

$$\begin{aligned} E_{f_X}[\ X\ ] &= \int_0^{\infty} x f_X(x)\ dx = \int_0^{\infty}\left\{\int_0^x dy\right\} f_X(x)\ dx = \int_0^{\infty}\left\{\int_y^{\infty} f_X(x)\ dx\right\}\ dy \\ &= \int_0^{\infty}(1 - F_X(y))\ dy \equiv \int_0^{\infty}(1 - F_X(x))\ dx \end{aligned}$$

$$\begin{aligned} E_{f_X}[\ X^r\ ] &= \int_0^{\infty} x^r f_X(x)\ dx = \int_0^{\infty}\left\{\int_0^x r y^{r-1}\ dy\right\} f_X(x)\ dx = \int_0^{\infty}\left\{\int_y^{\infty} f_X(x)\ dx\right\} r y^{r-1}\ dy \\ &= \int_0^{\infty}(1 - F_X(y)) r y^{r-1}\ dy \equiv \int_0^{\infty} r x^{r-1}(1 - F_X(x))\ dx \end{aligned}$$

Note: the exchange of order of integration is valid if we know that the expectation integral is finite. This regarded as a standard result for random variables taking only non-negative values, and also holds in the discrete case with integrals replaced by summations. The important thing is to remember the trick of introducing a second integral involving dummy variable y. The rest of the result follows after careful manipulation of the double integral.

Now, for a random variable that takes values on $\mathbb{R}$, we split the integral into two at the origin and proceed as above, as follows.

$$\begin{aligned}
E_{f_X}[\,X^r\,] &= \int_{-\infty}^{\infty} x^r f_X(x)\,dx = \int_{-\infty}^{0} x^r f_X(x)\,dx + \int_0^{\infty} x^r f_X(x)\,dx \\
&= \int_{-\infty}^{0} \left\{ \int_0^x r y^{r-1}\,dy \right\} f_X(x)\,dx + \int_0^{\infty} r x^{r-1}(1-F_X(x))\,dx \\
&= \int_{-\infty}^{0} \left\{ -\int_x^0 r y^{r-1}\,dy \right\} f_X(x)\,dx + \int_0^{\infty} (1-F_X(y)) r y^{r-1}\,dy \\
&= -\int_{-\infty}^{0} r y^{r-1} \left\{ \int_{-\infty}^{y} f_X(x)\,dx \right\} dy + \int_0^{\infty} (1-F_X(y)) r y^{r-1}\,dy \\
&= -\int_{-\infty}^{0} r y^{r-1} F_X(y)\,dy + \int_0^{\infty} (1-F_X(y)) r y^{r-1}\,dy
\end{aligned}$$

3. We have that

$$\begin{aligned}
\mathrm{E}_{f_2}[\,X_2^r\,] &= \int_0^{\infty} x^r f_2(x)\,dx = \int_0^{\infty} x^r \left[1+\sin(2\pi \log x)\right] f_1(x)\,dx \\
&= \int_0^{\infty} x^r f_1(x)\,dx + \int_0^{\infty} x^r \sin(2\pi\log x) f_1(x)\,dx \\
&= \mathrm{E}_{f_1}[\,X_1^r\,] + \int_0^{\infty} x^r \sin(2\pi\log x) c x^{-1} \exp\left\{-\frac{(\log x)^2}{2}\right\}\,dx \\
&= \mathrm{E}_{f_1}[\,X_1^r\,] + c\int_{-\infty}^{\infty} e^{rt}\sin(2\pi t)\exp\left\{-\frac{t^2}{2}\right\}\,dt \qquad \text{(putting } t=\log x\text{)} \\
&= \mathrm{E}_{f_1}[\,X_1^r\,] + c\exp\left\{r^2/2\right\}\int_{-\infty}^{\infty} \sin(2\pi t)\exp\left\{-\frac{(t-r)^2}{2}\right\}\,dt \qquad \text{(completing the square in } t\text{))} \\
&= \mathrm{E}_{f_1}[\,X_1^r\,] + c\exp\left\{r^2/2\right\}\int_{-\infty}^{\infty} \sin(2\pi(s+r))\exp\left\{-\frac{s^2}{2}\right\}\,ds \qquad \text{(putting } s=t-r\text{)} \\
&= \mathrm{E}_{f_1}[\,X_1^r\,] + c\exp\left\{r^2/2\right\}\int_{-\infty}^{\infty} \sin(2\pi s)\exp\left\{-\frac{s^2}{2}\right\}\,ds \;= \mathrm{E}_{f_1}[\,X_1^r\,]
\end{aligned}$$

as $\sin(2\pi(s+r)) = \sin(2\pi s)$ for $r = 1, 2, ...$, as the integrand is an integrable, odd function about zero.

The result follows after showing that the second integral is zero; it may not be obvious when you start the manipulation, but the $t = \log x$ substitution seems a natural first step - this has two advantages; first it gets rid of the awkward log terms and secondly it changes the range of integration to the whole real line leaving an integrand that looks more familiar and tractable. The next step of completing the square takes a little spotting, but also seems sensible to combine the exp terms. The remainder of the calculation is similar to the the Cauchy example from the lectures; here the integral is zero as the integrand is an integrable odd function.

4. (i) By integration, for $x \geq 0$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x \alpha^2 t \exp\{-\alpha t\}\ dt = [-\alpha t \exp\{-\alpha t\}]_0^x + \int_0^x \alpha \exp\{-\alpha t\}\ dt$$

$$= -\alpha x \exp\{-\alpha x\} + [-\exp\{-\alpha t\}]_0^x = 1 - (1 + \alpha x)\exp\{-\alpha x\}$$

Hence $\mathrm{P}[\ X \geq m\ ] = 1 - \mathrm{P}[\ X < m\ ] = 1 - F_X(m) = (1 + \alpha m)\exp\{-\alpha m\}$

(ii)

$$\mathrm{E}_{f_X}[\ X\ ] = \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_0^{\infty} x\alpha^2 x \exp\{-\alpha x\}\ dx = \left[-\alpha x^2 \exp\{-\alpha x\}\right]_0^{\infty} + \int_0^{\infty} 2x\alpha \exp\{-\alpha x\}\ dx$$

$$= 0 + \frac{2}{\alpha}\int_0^{\infty} x\alpha^2 \exp\{-\alpha x\}\ dx = \frac{2}{\alpha}$$

as the integrand is a pdf. Hence a change in the expectation to $2/\beta$ corresponds to a change from α to β in the pdf and cdf. Hence $\mathrm{P}[\ X \geq m\ ]$ changes to $(1 + \beta m)\exp\{-\beta m\}$.

5. The cdf of X, F_X is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x 4t^3\ dt = x^4 \qquad 0 < x < 1.$$

(a) $Y = X^4$, so $\mathbb{Y} = (0, 1)$, and from first principles, for $y \in \mathbb{Y}$,

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ X^4 \leq y\ ] = \mathrm{P}[\ X \leq y^{1/4}\ ] = F_X(y^{1/4}) = y \qquad \Longrightarrow f_Y(y) = 1 \qquad 0 < y < 1$$

(b) $W = e^X$, so $\mathbb{W} = (1, e)$, and from first principles, for $w \in \mathbb{W}$,

$$F_W(w) = \mathrm{P}[\ W \leq w\ ] = \mathrm{P}[\ e^X \leq w\ ] = \mathrm{P}[\ X \leq \log w\ ] = F_X(\log w) = (\log w)^4$$

$$\Longrightarrow \quad f_W(w) = \frac{4(\log w)^3}{w} \qquad 1 < w < e$$

(c) $Z = \log X$, so $\mathbb{Z} = (-\infty, 1)$, and from first principles, for $z \in \mathbb{Z}$,

$$F_Z(z) = \mathrm{P}[\ Z \leq z\ ] = \mathrm{P}[\ \log X \leq z\ ] = \mathrm{P}[\ X \leq e^z\ ] = F_X(e^z) = e^{4z} \Longrightarrow f_Z(z) = 4e^{4z} \qquad -\infty < z < 1$$

(d) $U = (X - 0.5)^2$, so $\mathbb{U} = (0, 0.25)$, and from first principles, for $u \in \mathbb{U}$,

$$F_U(u) = \mathrm{P}[\ U \leq u\ ] = \mathrm{P}[\ (X - 0.5)^2 \leq u\ ] = \mathrm{P}[\ -\sqrt{u} + 0.5 \leq X \leq \sqrt{u} + 0.5\ ]$$

$$= F_X(\sqrt{u} + 0.5) - F_X(-\sqrt{u} + 0.5) = (0.5 + \sqrt{u})^4 - (0.5 - \sqrt{u})^4$$

$$\Longrightarrow f_U(u) = \frac{2}{\sqrt{u}}\left[(0.5 + \sqrt{u})^3 + (0.5 - \sqrt{u})^3\right] = \frac{1 + 12u}{2\sqrt{u}} \qquad -0 < u < 0.25$$

To find the decreasing function H on $(0, 1)$; need $F_V(v) = v,\ 0 < v < 1$, that is, need

$$\mathrm{P}[\ V \leq v\ ] = \mathrm{P}[\ H(X) \leq v\ ] = v \Longrightarrow \mathrm{P}[\ X \geq H^{-1}(v)\ ] = v \Longrightarrow 1 - \mathrm{P}[\ X < H^{-1}(v)\ ] = v$$

$$\Longrightarrow \quad \left\{H^{-1}(v)\right\}^4 = 1 - v \text{ and hence } H(v) = 1 - v^4$$

6. We have $f_R(r) = 6r(1-r)$, for $0 < r < 1$, and hence

$$F_R(r) = r^2(3-2r) \qquad 0 < r < 1$$

Circumference: $Y = 2\pi R$, so $\mathbb{Y} = (0, 2\pi)$, and from first principles, for $y \in \mathbb{Y}$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}[\ Y \le y\ ] = \mathrm{P}[\ 2\pi R \le y\ ] = \mathrm{P}[\ R \le y/2\pi\ ] = F_R(y/2\pi) = \frac{3y^2}{4\pi^2} - \frac{2y^3}{8\pi^3} \\ \Longrightarrow f_Y(y) &= \frac{6y}{8\pi^3}(2\pi - y) \qquad 0 < y < 2\pi \end{aligned}$$

Area: $Z = \pi R^2$, so $\mathbb{Z} = (0, \pi)$, and from first principles, for $z \in \mathbb{Z}$, recalling that f_R is only positive when $0 < z < \pi$,

$$\begin{aligned} F_Z(z) &= \mathrm{P}[\ Z \le z\ ] = \mathrm{P}[\ \pi R^2 \le z\ ] = \mathrm{P}[\ R \le \sqrt{z/\pi}\ ] = F_R(z/2\pi) = \frac{3z}{\pi} - 2\left\{\frac{z}{\pi}\right\}^{3/2} \\ \Longrightarrow f_Z(z) &= 3\pi^{-3/2}(\sqrt{\pi} - \sqrt{z}) \qquad 0 < z < \pi. \end{aligned}$$

7. By integration

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt = \int_0^x \frac{\alpha}{\beta}\left(\frac{\beta}{\beta + t}\right)^{\alpha+1} dt = \left[-\left(\frac{\beta}{\beta+t}\right)^{\alpha}\ \right]_0^x = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha} \qquad x > 0.$$

If $Y = \log X$, then $\mathbb{Y} = \mathbb{R}$, and

$$\begin{aligned} F_Y(y) &= \mathrm{P}[\ Y \le y\ ] = \mathrm{P}[\ \log X \le y\ ] = \mathrm{P}[\ X \le e^y\ ] = F_X(e^y) = 1 - \left(1 + \frac{e^y}{\beta}\right)^{-\alpha} \\ \Longrightarrow f_Y(y) &= \frac{\alpha}{\beta} e^y \left(\frac{\beta}{\beta + e^y}\right)^{\alpha+1} \qquad y \in \mathbb{R} \end{aligned}$$

If $Z = \xi + \theta Y$, then $Y = (Z - \xi)/\theta$, so the density of Z can be found easily using transformation techniques

$$f_Z(z) = \frac{\alpha}{\beta} e^{(z-\xi)/\theta} \left(\frac{\beta}{\beta + e^{(z-\xi)/\theta}}\right)^{\alpha+1} \frac{1}{\theta} \qquad z \in \mathbb{R}$$