M2S1 : ASSESSED COURSEWORK 1 : SOLUTIONS

1. (a) We need that the sum over all possible locations is 1. Thus

$$\sum_{(x,y) \text{ on the lattice}} \frac{c(\gamma,\phi)\,\gamma^x}{x!\phi^y} = 1 \qquad \Longrightarrow \qquad c(\gamma,\phi)\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{\gamma^x}{x!\phi^y} = 1.$$

Thus

$$[c(\gamma,\phi)]^{-1} = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{\gamma^x}{x!\phi^y} = \left\{\sum_{x=0}^{\infty}\frac{\gamma^x}{x!}\right\}\left\{\sum_{y=0}^{\infty}\phi^{-y}\right\} = e^{\gamma}\frac{1}{1-\dfrac{1}{\phi}} = \frac{\phi e^{\gamma}}{\phi-1}$$

summing an exponential series and a geometric series respectively (note $\frac{1}{\phi} < 1$ as $\phi > 1$). Thus

$$c(\gamma,\phi) = \frac{(\phi-1)\,e^{-\gamma}}{\phi}$$

so that the mass function is given by

$$\frac{e^{-\gamma}\gamma^x(\phi-1)}{x!\phi^{y+1}} \qquad x = 0,1,2,3,...,y = 0,1,2,3,...$$

[3 MARKS]

(b) For $x = 1$, $y = 1$, with $\gamma = 1$ and $\phi = 3$

$$\frac{e^{-\gamma}\gamma^x(\phi-1)}{x!\phi^{y+1}} = \frac{e^{-1}1^1(3-1)}{1!3^2} = 0.0818$$

[2 MARKS]

(c) The probability that the particle lies on the line $y = 1$ is given by the Theorem of Total Probability, using a partition into the different possible x values. Summing out the probabilities of points on the lattice that lie on that line,we have

$$\sum_{x=0}^{\infty}\frac{\gamma^x(\phi-1)}{x!\phi^{y+1}} = \frac{(\phi-1)}{\phi^{y+1}}\left\{\sum_{x=0}^{\infty}\frac{e^{-\gamma}\gamma^x}{x!}\right\} = \frac{(\phi-1)}{\phi^{y+1}}$$

when $y = 1$, that is

$$\frac{(\phi-1)}{\phi^2} = \frac{3-1}{3^2} = \frac{2}{9}$$

[2 MARKS]

(d) The particle lies no further than two units away from $(0,0)$ if it lies at the points

$$(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$$

so summing up the probabilities over these six points gives the answer 0.8039, so the probability that it lies further away than that is

$$1 - 0.8039 = 0.1961$$

[3 MARKS]

2.(a) To find c: note that we need

$$\lim_{x\to\infty} F_X(x) = 1$$

so, direct from the form of F_X, we have that

$$\frac{c(\mu,\sigma,\alpha)}{\left\{1+\exp\left\{-\left(\dfrac{x-\mu}{\sigma}\right)\right\}\right\}^{\alpha}} \to 1 \qquad \text{as } x\to\infty$$

so we deduce that $c(\mu,\sigma,\alpha)=1$.

[2 MARKS]

(b) By differentiation

$$f_X(x) = \frac{\alpha}{\sigma}\frac{\exp\left\{-\left(\dfrac{x-\mu}{\sigma}\right)\right\}}{\left\{1+\exp\left\{-\left(\dfrac{x-\mu}{\sigma}\right)\right\}\right\}^{\alpha+1}} \qquad x\in\mathbb{R}$$

[2 MARKS]

(c) $F_X(x_M)=0.5$ implies that x_M is the solution to

$$\frac{1}{\left\{1+\exp\left\{-\left(\dfrac{x_M-\mu}{\sigma}\right)\right\}\right\}^{\alpha}} = \frac{1}{2} \Leftrightarrow x_M = \mu - \sigma\log\left(2^{1/\alpha}-1\right).$$

Then f_X is symmetric about x_M if, for $x>0$

$$f_X(x_M-x) = f_X(x_M+x).$$

Now

$$\begin{aligned} f_X(x_M-x) &= f_X\left(\mu-\sigma\log\left(2^{1/\alpha}-1\right)-x\right) = \frac{\alpha}{\sigma}\frac{\exp\left\{\log\left(2^{1/\alpha}-1\right)+x\right\}}{\left\{1+\exp\left\{\log\left(2^{1/\alpha}-1\right)+x\right\}\right\}^{\alpha+1}} \\ &= \frac{\alpha}{\sigma}\frac{\left(2^{1/\alpha}-1\right)e^{x}}{\left\{1+\left(2^{1/\alpha}-1\right)e^{x}\right\}^{\alpha+1}} \neq \frac{\alpha}{\sigma}\frac{\left(2^{1/\alpha}-1\right)e^{-x}}{\left\{1+\left(2^{1/\alpha}-1\right)e^{-x}\right\}^{\alpha+1}} = f_X(x_M+x) \end{aligned}$$

unless $\alpha=1$. This is the only case that gives symmetry about x_M.

[1 MARK]

(d) The range of the transformed variable is $\mathbb{Y}\equiv(0,1)$, as F_X always returns a number between 0 and 1 as it is a cdf. The answer $\mathbb{Y}\equiv[0,1]$ is also acceptable.

[1 MARK]

From first principles, for $y\in(0,1)$,

$$F_Y(y) = P[Y\le y] = P[F_X(X)\le y] = P\left[X\le F_X^{-1}(y)\right] = F_X\left(F_X^{-1}(y)\right) = y$$

as the transformation $g(t)=F_X(t)$ is monotone and increasing (in this case, and in general for continuous random variables).

[3 MARKS]

Thus by differentiation

$$f_Y(y) = 1 \qquad 0<y<1$$

and zero otherwise.

[1 MARK]