M2S1 : EXERCISE SHEET 6 : SOLUTIONS

1. As $X_1, ..., X_n \sim Poisson(\lambda)$, and given that $T_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then using elementary properties of expectations, we have

$$\mathrm{E}_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n} \lambda = \lambda$$

so that T_1 is an *unbiased* estimator of λ. Furthermore

$$T_2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\left(\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} X_i\right)^2$$

From properties of expectations, variances, and the Poisson distribution,

$$\mathrm{E}_{f_X}\left[X^2\right] = \mathrm{Var}_{f_X}[X] + \left\{\mathrm{E}_{f_X}[X]\right\}^2 = \lambda + \lambda^2 = \lambda(\lambda+1)$$

Now, from properties of independent Poisson random variables

$$Y_n = \sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$$

so therefore, taking expectations in the above

$$\mathrm{E}_{f_{T_2}}[T_2] = \frac{1}{n-1}\sum_{i=1}^{n}\mathrm{E}_{f_{X_i}}\left[X_i^2\right] - \frac{1}{n(n-1)}\mathrm{E}_{f_{Y_n}}\left[Y_n^2\right] = \frac{1}{n-1}\sum_{i=1}^{n}\lambda(\lambda+1) - \frac{1}{n(n-1)}n\lambda(n\lambda+1)$$

$$= \frac{n}{n-1}\lambda(\lambda+1) - \frac{1}{n-1}\lambda(n\lambda+1) = \frac{1}{n-1}\left[n\lambda(\lambda+1) - \lambda(n\lambda+1)\right]$$

$$= \frac{1}{n-1}\left[n\lambda^2 + n\lambda - n\lambda^2 + \lambda\right] = \lambda$$

2. From the theorem in lectures, and by properties of the Gamma distribution, we can write

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2_{n-1} \equiv Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right) \Longrightarrow V_n = \sum_{i=1}^{n-1} X_i$$

where $X_i \sim \chi^2_1 \equiv Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, so that, using the Gamma expectation and variance results

$$\mathrm{E}_{f_{X_i}}[X_i] = \frac{1/2}{1/2} = 1 = \mu \qquad \mathrm{Var}_{f_{X_i}}[X_i] = \frac{1/2}{(1/2)^2} = 2 = \sigma^2$$

say. Hence, by the Central Limit Theorem,

$$\frac{V_n - (n-1)\mu}{\sqrt{(n-1)\sigma^2}} = \frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0,1)$$

Hence, substituting in the definition for V_n,

$$\frac{\frac{(n-1)s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}\left(s_n^2 - \sigma^2\right)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1)$$

and finally, by a location/scale transformation to $Z_n = \sigma^2 + \frac{\sigma^2\sqrt{2}}{\sqrt{n-1}}Z$, we have

$$s_n^2 \xrightarrow{d} Z_n \sim N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

3. $X_1, ..., X_n \sim Gamma(\alpha, \beta)$ so that $\mathrm{E}_{f_{X_i}}[X_i] = \alpha/\beta$ and $\mathrm{Var}_{f_{X_i}}[X_i] = \alpha/\beta^2$ so that

$$E_{f_{X_i}}\left[X_i^2\right] = \mathrm{Var}_{f_{X_i}}[X_i] + \left\{\mathrm{E}_{f_{X_i}}[X_i]\right\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha+1)}{\beta^2}$$

Hence for the method of moments estimators $\hat{\alpha}_{MM}$ and $\hat{\beta}_{MM}$, need to solve the following pairs of equations

FIRST MOMENT	Solve	$\frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} = \frac{\alpha}{\beta}$
SECOND MOMENT	Solve	$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = (\overline{x})^2 + S^2 = \frac{\alpha(\alpha+1)}{\beta^2}$

where

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\overline{x})^2$$

Elementary algebra gives

$$\hat{\alpha}_{MM} = \frac{(\overline{x})^2}{S^2} \qquad \hat{\beta}_{MM} = \frac{\overline{x}}{S^2}$$

4. Follow the four step procedure that can be summarized as follows: for an observed random sample $x_1, ..., x_n$ from a distribution represented by mass/density function $f_X(x;\theta)$

STEP 1 : Form the likelihood function $L(\theta)$

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

STEP 2 Take (natural) log to obtain $\log L(\theta)$

STEP 3: Find the value of θ at which $\log L(\theta)$ (and hence $L(\theta)$) is maximized within the parameter space Θ by differentiation

STEP 4: Check the **maximum** value has been found.

Formally, we define the **maximum likelihood estimate** of θ, $\hat{\theta}_{ML}$, as

$$\hat{\theta}_{ML} = \arg\max_{\theta} L(\theta)$$

Hence, for the $Poisson(\lambda)$ case

STEP 1 $$L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\left(\prod_{i=1}^{n} x_i!\right)}$$

STEP 2 $$\log L(\lambda) = -\sum_{i=1}^{n} \log x_i! - n\lambda + \left(\sum_{i=1}^{n} x_i\right) \log \lambda$$

STEP 3 $$\frac{d}{d\lambda}\{\log L(\lambda)\} = -n + \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

STEP 4 $$\frac{d^2}{d\lambda^2}\{\log L(\lambda)\} = -\left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda^2} < 0 \qquad \text{for all } \lambda$$

Therefore

$$\text{ESTIMATE} : \hat{\lambda}_{ML} = \overline{x} \qquad\qquad \text{ESTIMATOR: } \overline{X}$$

and from question 1 on this sheet, we know that if $T_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ then $E_{f_{T_1}}[T_1] = \lambda$ and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \qquad\qquad \text{so that} \qquad\qquad \lambda = -\log \tau$$

and we can reformulate the likelihood in terms of τ, giving

$$\log L(\tau) = -\sum_{i=1}^{n} \log x_i! + n \log \tau + \left(\sum_{i=1}^{n} x_i\right) \log(-\log \tau)$$

and

$$\frac{d}{d\lambda}\{\log L(\tau)\} = \frac{\left(\sum_{i=1}^{n} x_i\right)}{-\tau \log \tau} + \frac{n}{\tau} = 0 \Longrightarrow \qquad \hat{\tau}_{ML} = \exp\left\{-\frac{1}{n}\sum_{i=1}^{n} x_i\right\} = \exp\{-\overline{x}\}$$

which can be shown to be the value that maximizes the likelihood, so that

$$\hat{\tau}_{ML}(\lambda) = \tau\left(\hat{\lambda}_{ML}\right)$$

5. (i) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

STEP 2 $$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n}\log x_i} \qquad \text{ESTIMATOR:} \; -\frac{n}{\sum_{i=1}^{n}\log X_i}$$

(ii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} (\theta+1)\, x_i^{-(\theta+2)} = \frac{(\theta+1)^n}{\left(\prod_{i=1}^{n} x_i\right)^{(\theta+2)}}$$

STEP 2 $$\log L(\theta) = n\log(\theta+1) - (\theta+2)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta+1} - \sum_{i=1}^{n}\log x_i = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{(\theta+1)^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1 \qquad \text{ESTIMATOR:} \; = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1$$

(iii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n}\left(\prod_{i=1}^{n} x_i\right)\exp\left\{-\theta\sum_{i=1}^{n} x_i\right\}$$

STEP 2 $$\log L(\theta) = 2n\log\theta + \sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^{n} x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{2n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^{n} x_i} \qquad \text{ESTIMATOR:} = \frac{2n}{\sum_{i=1}^{n} X_i}$$

(iv) Because of the constraint in the pdf that $x \le \theta$

STEP 1 $$L(\theta) \quad \prod_{i=1}^{n} f_X(x_i;\theta) == \begin{cases} \prod_{i=1}^{n} 2\theta^2 x_i^{-3} = 2^n\theta^{2n}\left(\prod_{i=1}^{n} x_i^{-3}\right) & \theta \le x_1, ..., x_n \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log L(\theta) = n\log 2 + 2n\log\theta - 3\sum_{i=1}^{n}\log x_i$$

At this point we note that the likelihood is monotonically increasing in θ, and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \le x_1, ..., x_n$ is still satisfied, hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \min\{x_1, ..., x_n\} \qquad \text{ESTIMATOR: } \min\{X_1, ..., X_n\}$$

(v) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n}\left(\frac{\theta}{2}\right)\exp\{-\theta|x_i|\} = 2^{-n}\theta^n \exp\left\{-\theta\sum_{i=1}^{n}|x_i|\right\}$$

STEP 2 $$\log L(\theta) = -n\log 2 + n\log\theta - \theta\sum_{i=1}^{n}|x_i|$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta} - \sum_{i=1}^{n}|x_i| = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}|x_i|}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} |x_i|} \qquad \text{ESTIMATOR}: \frac{n}{\sum_{i=1}^{n} |x_i|}$$

(vi) Because of the constraint in the pdf that $\theta_1 \leq x \leq \theta_2$

STEP 1 $$L(\theta_1, \theta_2) \quad \prod_{i=1}^{n} f_X(x_i; \theta) = \begin{cases} \prod_{i=1}^{n} \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n} & \theta_1 \leq x_1, ..., x_n \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log L(\theta_1, \theta_2) \quad = -n \log (\theta_2 - \theta_1)$$

At this point we note that the likelihood is monotonically increasing in $\theta_1$and monotonically decreasing in θ_2, and hence the likelihood is maximized when θ_1 is as **large** as possible and when θ_2 is as **small** as possible, but so that the constraint $\theta_1 \leq x_1, ..., x_n \leq \theta_2$ (and $\theta_2 \geq \theta_1$) is still satisfied, hence

$$\text{ESTIMATE}: \begin{array}{ll} \hat{\theta}_{1ML} & = \min\{x_1, ..., x_n\} \\ \hat{\theta}_{2ML} & = \max\{x_1, ..., x_n\} \end{array} \qquad \text{ESTIMATOR:} \begin{array}{ll} \theta_1 & \min\{X_1, ..., X_n\} \\ \theta_2 & \max\{X_1, ..., X_n\} \end{array}$$

(vii) Noting the constraint in the pdf that $x \geq \theta_2$, we have

STEP 1 $$L(\theta_1, \theta_2) \quad = \prod_{i=1}^{n} f_X(x_i; \theta) = \prod_{i=1}^{n} \theta_1 \theta_2^{\theta_1} x_i^{-(\theta_1+1)} = \theta_1^n \theta_2^{n\theta_1} \left(\prod_{i=1}^{n} x_i \right)^{-(\theta_1+1)}$$

$$0 \leq x_1, ..., x_n \leq \theta_2$$

STEP 2 $$\log L(\theta_1, \theta_2) \quad = n \log \theta_1 + n\theta_1 \log \theta_2 - (\theta_1 + 1) \sum_{i=1}^{n} \log x_i$$

STEP 3 $$\frac{\partial}{\partial \theta_1} \{\log L(\theta_1, \theta_2)\} \quad = \frac{n}{\theta_1} + n \log \theta_2 - \sum_{i=1}^{n} \log x_i = 0$$

$$\implies \hat{\theta}_{1ML} = \frac{n}{\sum_{i=1}^{n} \log x_i - n \log \hat{\theta}_{2ML}}$$

$$\frac{\partial}{\partial \theta_2} \{\log L(\theta_1, \theta_2)\} \quad = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs

when θ_2 is as **large** as possible, that is, when $\hat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$ .Hence

$$\hat{\theta}_{1ML} = \frac{n}{\left[\sum_{i=1}^{n} \log x_i - n \log\{\min\{x_1, ..., x_n\}\}\right]}$$

ESTIMATES

$$\hat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$$

$$\theta_1 \quad \frac{n}{\left[\sum_{i=1}^{n} \log x_i - n \log\{\min\{X_1, ..., X_n\}\}\right]}$$

ESTIMATORS:

$$\theta_2 \quad \min\{X_1, ..., X_n\}$$

6. Noting the constraint in the pdf that $x \geq \eta$, we have

STEP 1 $$L(\lambda, \eta) = \prod_{i=1}^{n} f_X(x_i; \theta) = \prod_{i=1}^{n} \lambda \exp\{-\lambda(x_i - \eta)\} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n} (x_i - \eta)\right\}$$
$\eta \leq x_1, ..., x_n$, zero otherwise

STEP 2 $$\log L(\lambda, \eta) = n \log \lambda - \lambda \sum_{i=1}^{n} (x_i - \eta) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i + n\lambda\eta$$

STEP 3 $$\frac{\partial}{\partial \lambda}\{\log L(\lambda, \eta)\} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - n\eta = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i - n\hat{\eta}}$$

$$\frac{\partial}{\partial \eta}\{\log L(\lambda, \eta)\} = n\lambda$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when η is as **large** as possible, that is, when $\hat{\eta} = \min\{x_1, ..., x_n\}$ .Hence

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i - n \min\{x_1, ..., x_n\}} = \frac{n}{\sum_{i=1}^{n} (x_i - \min\{x_1, ..., x_n\})}$$

ESTIMATES

$$\hat{\eta} = \min\{x_1, ..., x_n\}$$

$$\lambda = \frac{n}{\sum_{i=1}^{n} (X_i - \min\{X_1, ..., X_n\})}$$

ESTIMATORS:

$$\eta \quad \min\{X_1, ..., X_n\}$$

7. $X_1, ..., X_n \sim Exponential(1/\theta)$ so that $\mathrm{E}_{f_{X_i}}[X_i] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^{n} X_i \sim Gamma\left(n, \frac{1}{\theta}\right) \qquad \Longrightarrow \qquad E_{f_X}[X] = \frac{n}{\frac{1}{\theta}} = n\theta$$

so that if $T_1 = \overline{X} = \frac{1}{n}X$ then

$$E_{f_{T_1}}[T_1] = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta} \qquad y > 0$$

so that $Y_1 \sim Exponential\left(\frac{n}{\theta}\right)$. Hence if $T_2 = nY_1$ then

$$E_{f_{Y_1}}[Y_1] = \frac{\theta}{n} \qquad \therefore \qquad E_{f_{T_2}}[T_2] = n\frac{\theta}{n} = \theta$$

and hence T_2 is an unbiased estimator of θ.

$$f_X(x) = \frac{1}{2} \qquad \theta - 1 \leq x \leq \theta + 1 \qquad\qquad F_X(x) = \frac{x - (\theta - 1)}{2} = \frac{x - \theta + 1}{2} \qquad \theta - 1 \leq x \leq \theta + 1$$

$\mathrm{E}_{f_{X_i}}[X_i] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \overline{X}$

$$\mathrm{E}_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$and $Y_n = \max\{X_1, ..., X_n\}$, then previous order statistics results give that

$$f_{Y_1}(y) = nf_X(y)\{1 - F_X(y)\}^{n-1} = n\frac{1}{2}\left\{1 - \frac{y - (\theta - 1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1 + \theta - y}{2}\right\}^{n-1} \qquad \theta - 1 \leq y \leq \theta + 1$$

and

$$f_{Y_n}(y) = nf_X(y)\{F_X(y)\}^{n-1} = n\frac{1}{2}\left\{\frac{y - (\theta - 1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1 - \theta + y}{2}\right\}^{n-1} \qquad \theta - 1 \leq y \leq \theta + 1$$

For the expectations,

$$\begin{aligned}
\mathrm{E}_{f_{Y_1}}[Y_1] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1 + \theta - y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 ((1 + \theta) - 2t)\, t^{n-1}\, 2\, dt \qquad\qquad \text{setting } t = (1 + \theta - y)/2 \therefore y = (1 + \theta) - 2t \\
&= (1 + \theta)\int_0^1 nt^{n-1}\, dt - 2n\int_0^1 t^n dt \\
&= (1 + \theta) - \frac{2n}{n + 1}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_{f_{Y_n}}[Y_n] &= \int_{\theta-1}^{\theta+1} y \frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 (2t-(1-\theta))\, t^{n-1}\, 2\, dt \qquad \text{setting } t=(1-\theta+y)/2 \therefore y=2t-(1-\theta) \\
&= 2\int_0^1 nt^n\, dt - (1-\theta)\int_0^1 nt^{n-1} dt \\
&= \frac{2n}{n+1} - (1-\theta)
\end{aligned}$$

so that if $M=(Y_1+Y_n)/2$ then by properties of expectations

$$\mathrm{E}_{f_M}[M] = \frac{1}{2}\mathrm{E}_{f_{Y_1}}[Y_1] + \frac{1}{2}\mathrm{E}_{f_{Y_n}}[Y_n] = \left[\frac{1}{2}(1+\theta) - \frac{n}{n+1}\right] + \left[\frac{n}{n+1} - \frac{1}{2}(1-\theta)\right] = \theta$$

and hence M is an unbiased estimator for θ

9 For the m.l.e. for $\tau = 1/\theta$, write the likelihood in terms of τ

STEP 1
$$\begin{aligned}
L(\tau) &= \prod_{i=1}^n f_X(x_i;\theta) = \prod_{i=1}^n (1-\theta)^{x_i-1}\,\theta = \prod_{i=1}^n \left(\frac{\tau-1}{\tau}\right)^{x_i-1}\frac{1}{\tau} \\
&= \frac{(\tau-1)^{\sum_{i=1}^n x_i - n}}{\tau^{\sum_{i=1}^n x_i}}
\end{aligned}$$

STEP 2
$$\log L(\tau) = \left(\sum_{i=1}^n x_i - n\right)\log(\tau-1) - \left(\sum_{i=1}^n x_i\right)\log\tau$$

STEP 3
$$\frac{d}{d\theta}\{\log L(\tau)\} = \frac{\left(\sum_{i=1}^n x_i - n\right)}{\tau-1} - \frac{\sum_{i=1}^n x_i}{\tau} = 0 \qquad \Longrightarrow \qquad \hat{\tau}_{ML} = \frac{1}{n}\sum_{i=1}^n x_i = \overline{x}$$

STEP 4
$$\begin{aligned}
\frac{d^2}{d\theta^2}\{\log L(\tau)\} &= -\frac{\left(\sum_{i=1}^n x_i - n\right)}{(\tau-1)^2} + \frac{\sum_{i=1}^n x_i}{\tau^2} = -\frac{(\overline{x}-1)}{n(\tau-1)^2} + \frac{\overline{x}}{n\tau^2} \\
&= -\frac{1}{n\tau^2(\tau-1)^2}\left[\tau^2(\overline{x}-1) + \overline{x}(\tau-1)^2\right] < 0 \qquad \text{at } \tau = \hat{\tau} = \overline{x}
\end{aligned}$$

and hence then ML estimator is $T=\overline{X}$. From elementary properties of expectations and variances, we have that

$$\mathrm{E}_{f_T}[T] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^n \frac{(1-\theta)}{\theta} = \frac{1}{n}\frac{n}{\theta} = \frac{1}{\theta} = \tau$$

and

$$\mathrm{Var}_{f_T}[T] = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}_{f_{X_i}}[X_i] = \frac{1}{n^2}\sum_{i=1}^n \frac{(1-\theta)}{\theta^2} = \frac{1}{n^2}\frac{n(1-\theta)}{\theta^2} = \frac{(1-\theta)}{n\theta^2} = \frac{(\tau-1)}{n\tau}$$

so T is an unbiased estimator for with variance $(\tau-1)/n\tau$.

Now, for the Cramer-Rao bound,

$$\frac{\partial}{\partial\theta}\left\{\log f_X(x;\theta)\right\} = \frac{\partial}{\partial\theta}\left\{(x-1)\log(1-\theta)+\log\theta\right\} = \frac{(x-1)}{(1-\theta)}+\frac{x}{\theta} = \frac{\left(x-\frac{1}{\theta}\right)}{(\theta-1)}$$

and hence

$$\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\left\{\log f_X(X;\theta)\right\}\right\}^2\right] = \mathrm{E}_{f_X}\left[\left\{\frac{\left(X-\frac{1}{\theta}\right)}{(\theta-1)}\right\}^2\right] = \frac{1}{(\theta-1)^2}\mathrm{E}_{f_X}\left[\left(X-\frac{1}{\theta}\right)^2\right] = \frac{1}{(\theta-1)^2}\mathrm{Var}_{f_X}[X]$$

as $\mathrm{E}_{f_X}\left[\left(X-\frac{1}{\theta}\right)^2\right] = \mathrm{E}_{f_X}\left[\left(X-\mathrm{E}_{f_X}[X]\right)^2\right]$. Hence

$$\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\left\{\log f_X(X;\theta)\right\}\right\}^2\right] = \frac{1}{(\theta-1)^2}\mathrm{Var}_{f_X}[X] = \frac{1}{(\theta-1)^2}\frac{(1-\theta)}{\theta^2} = \frac{1}{(1-\theta)\theta^2}$$

and hence the lower bound on the variance for unbiased estimators is

$$\frac{\left[\tau'(\theta)\right]^2}{n\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\left\{\log f_X(X;\theta)\right\}\right\}^2\right]} = \frac{\left[-\frac{1}{\theta^2}\right]^2}{n\frac{1}{(1-\theta)\theta^2}} = \frac{(1-\theta)}{n\theta^2}$$

But, from above, this is the variance of the estimator T, so T is UMVUE.

10. For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^n f_X(x_i;\theta) = \prod_{i=1}^n\left(\frac{1}{2\pi\theta}\right)^{1/2}\exp\left\{-\frac{x_i^2}{2\theta}\right\} = \left(\frac{1}{2\pi}\right)^{n/2}\theta^{-n/2}\exp\left\{-\frac{1}{2\theta}\sum_{i=1}^n x$$

STEP 2 $$\log L(\theta) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^n x_i^2$$

STEP 3 $$\frac{d}{d\theta}\left\{\log L(\theta)\right\} = -\frac{n}{2\theta}+\frac{1}{2\theta^2}\sum_{i=1}^n x_i^2 = 0 \qquad \Longrightarrow \qquad \hat\theta_{ML} = \frac{1}{n}\sum_{i=1}^n x_i^2$$

STEP 4 $$\frac{d^2}{d\theta^2}\left\{\log L(\theta)\right\} = \frac{n}{2\theta^2}-\frac{1}{\theta^3}\sum_{i=1}^n x_i^2 = \frac{n}{2\theta^3}\left[\theta-2\sum_{i=1}^n x_i^2\right] < 0 \qquad \text{at } \theta=\hat\theta_{ML}$$

Hence the estimator is

$$T = \frac{1}{n}\sum_{i=1}^n X_i^2$$

which can be re-written

$$T = \frac{\theta}{n}\sum_{i=1}^n\left(\frac{X_i}{\sqrt{\theta}}\right)^2 = \frac{\theta}{n}\sum_{i=1}^n Y_i$$

but $X_i \sim N(0,\theta) \Longrightarrow X_i/\sqrt{\theta} \sim N(0,1) \Longrightarrow Y_i = X_i^2/\theta \sim \chi_1^2 \equiv Gamma\left(\frac{1}{2},\frac{1}{2}\right)$ and hence

$$\mathrm{E}_{f_T}[T] = \frac{\theta}{n}\sum_{i=1}^{n}\mathrm{E}_{f_{Y_i}}[Y_i] = \frac{\theta}{n}\sum_{i=1}^{n}1 = \theta$$

and hence T is an unbiased estimator for θ. Also,

$$\mathrm{Var}_{f_T}[T] = \left(\frac{\theta}{n}\right)^2\sum_{i=1}^{n}\mathrm{Var}_{f_{Y_i}}[Y_i] = \left(\frac{\theta}{n}\right)^2 \times n \times 2 = \frac{2\theta^2}{n}$$

Now, for the Cramer-Rao bound,

$$\frac{\partial}{\partial\theta}\{\log f_X(x;\theta)\} = \frac{\partial}{\partial\theta}\left\{-\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta - \frac{1}{2\theta}x^2\right\} = -\frac{1}{2\theta} + \frac{1}{2\theta^2}x^2 = \frac{\left(x^2-\theta\right)}{2\theta^2}$$

and hence

$$\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\{\log f_X(X;\theta)\}\right\}^2\right] = \mathrm{E}_{f_X}\left[\left\{\frac{\left(X^2-\theta\right)}{2\theta^2}\right\}^2\right] = \frac{1}{4\theta^4}\mathrm{E}_{f_X}\left[\left(X^2-\theta\right)^2\right] = \frac{1}{4\theta^4}\mathrm{Var}_{f_X}\left[X^2\right]$$

as $\mathrm{E}_{f_X}\left[\left(X^2-\theta\right)^2\right]$ =$\mathrm{E}_{f_X}\left[\left(X^2-\mathrm{E}_{f_X}\left[X^2\right]\right)^2\right]$. Now we have

$$\frac{X^2}{\theta} \sim Gamma\left(\frac{1}{2},\frac{1}{2}\right) \Longleftrightarrow X^2 \sim Gamma\left(\frac{1}{2},\frac{1}{2\theta}\right) \therefore \mathrm{Var}_{f_X}\left[X^2\right] = \frac{\frac{1}{2}}{\left(\frac{1}{2\theta}\right)^2} = 2\theta^2$$

Hence

$$\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\{\log f_X(X;\theta)\}\right\}^2\right] = \frac{1}{4\theta^4}\mathrm{Var}_{f_X}\left[X^2\right] = \frac{1}{4\theta^4}2\theta^2 = \frac{1}{2\theta^2}$$

and hence the lower bound on the variance for unbiased estimators is

$$\frac{\left[\tau'(\theta)\right]^2}{n\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\{\log f_X(X;\theta)\}\right\}^2\right]} = \frac{1^2}{n\frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

But, from above, this is the variance of the estimator T, so T is UMVUE.

11. (i) For $\lambda > 0$

STEP 1 $$L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n}\frac{\lambda^2}{\Gamma(2)}x_i\exp\{-\lambda x_i\} = \frac{\lambda^{2n}}{\{\Gamma(2)\}^n}\left(\prod_{i=1}^{n}x_i\right)\exp\left\{-\lambda\sum_{i=1}^{n}x_i\right.$$

STEP 2 $$\log L(\lambda) = 2n\log\lambda - n\log\Gamma(2) + \sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\lambda)\} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{2n}{\sum_{i=1}^{n}x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\lambda)\} = -\frac{2n}{\lambda^2} < 0 \qquad \text{at } \lambda = \hat{\lambda}_{ML}$$

Hence the estimator is $2n/\left(\sum_{i=1}^{n} X_i\right)$

(ii) The result from q. 4 can be generalized so that for any function $\tau = \tau(\lambda)$

$$\hat{\tau}_{ML}(\lambda) = \tau\left(\hat{\lambda}_{ML}\right)$$

which is referred to as the **invariance** property of maximum likelihood estimators. Thus we must have that the ML estimator of $\tau = 1/\lambda$ is

$$T = \frac{1}{2n}\sum_{i=1}^{n} X_i$$

Now, using mgfs, it is straightforward to show that

$$\sum_{i=1}^{n} X_i \sim Gamma(2n, \lambda) \qquad \therefore \qquad T = \frac{1}{2n}\sum_{i=1}^{n} X_i \sim Gamma(2n, 2n\lambda)$$

so that

$$\mathrm{E}_{f_T}[T] = \frac{2n}{2n\lambda} = \frac{1}{\lambda} \qquad \mathrm{Var}_{f_T}[T] = \frac{2n}{(2n\lambda)^2} = \frac{1}{2n\lambda^2} \qquad \mathrm{E}_{f_T}\left[T^2\right] = \frac{1}{2n\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2n+1}{2n\lambda^2}$$

(iii) Using the Central Limit Theorem we have that

$$\frac{\sum_{i=1}^{n} X_i - \frac{2n}{\lambda}}{\sqrt{\frac{2n}{\lambda^2}}} \xrightarrow{d} Z \sim N(0,1)$$

so that, dividing through by $2n$

$$\frac{T - \frac{1}{\lambda}}{\sqrt{\frac{1}{2n\lambda^2}}} \xrightarrow{d} Z \sim N(0,1)$$

and hence, approximately

$$T \sim N\left(\frac{1}{\lambda}, \frac{1}{2n\lambda^2}\right)$$

and as $n \longrightarrow \infty$ then variance tends to zero, and hence for $\varepsilon > 0$

$$\mathrm{P}\left[\left|T - \frac{1}{\lambda}\right| < \varepsilon\right] \longrightarrow 1 \therefore T \xrightarrow{p} \frac{1}{\lambda}$$