M2S1 : EXERCISES 1 : SOLUTIONS

1. Let H_n be the event that n tosses result in an even number of heads. Conditioning on the result after $n-1$ tosses, and using the Theorem of Total Probability

$$\mathrm{P}(H_n) = \mathrm{P}(H_n|H_{n-1})\mathrm{P}(H_{n-1}) + \mathrm{P}(H_n|H_{n-1}^{'})\mathrm{P}(H_{n-1}^{'})$$

Therefore,

$$p_n = (1-p)p_{n-1} + p(1-p_{n-1}) = p + (1-2p)p_{n-1}$$

Now, to find a solution to this difference equation, propose $p_n = A + B\lambda^n$ for all $n \geq 0$. Then

$$\left.\begin{array}{ll} n=0 & p_0 = A+B = 1 \\ \\ n \geq 1 & p_n = A + B\lambda^n = p + (1-2p)(A + B\lambda^{n-1}) \end{array}\right\} \Longrightarrow \lambda = (1-2p), A = B = \frac{1}{2},\ p_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n.$$

If $p < 1/2$, $(1-2p) > 0$, so $p_n > 1/2$ for all n.

As $n \longrightarrow \infty$,

$$p_n \longrightarrow \begin{cases} 1/2 & 0 < p < 1 \\ 1 & p = 0 \end{cases}$$

and if $p = 1$ no limit exists.

2. Let N_i be the event that i dice are used, and let S_j be the event that the sum is j, for $i, j = 1, 2, 3, ...$ Let E be the event that the number of dice is even. Then $E = \bigcup_{i=1}^{\infty} N_{2i}$.

(i) Using conditional probability definition

$$\mathrm{P}(S_4|E) = \frac{\mathrm{P}(S_4 \cap E)}{\mathrm{P}(E)} = \frac{\mathrm{P}\left(S_4 \cap \bigcup_{i=1}^{\infty} N_{2i}\right)}{\mathrm{P}(E)} = \frac{\mathrm{P}\left(\bigcup_{i=1}^{\infty} (S_4 \cap N_{2i})\right)}{\mathrm{P}(E)} = \frac{\mathrm{P}(S_4 \cap N_2) + \mathrm{P}(S_4 \cap N_4)}{\mathrm{P}(E)}$$

as $\mathrm{P}(S_4 \cap N_{2i}) = 0$ for $i = 3, 4, ...$ Now

$$\begin{aligned} \mathrm{P}(S_4 \cap N_2) + \mathrm{P}(S_4 \cap N_4) &= \mathrm{P}(S_4|N_2)\mathrm{P}(N_2) + \mathrm{P}(S_4|N_4)\mathrm{P}(N_4) \\ &= \frac{3}{36}\frac{1}{4} + \frac{1}{6^4}\frac{1}{2^4} = \frac{3}{144} + \frac{1}{20376} = \frac{433}{20736} \\ \mathrm{P}(E) &= \sum_{i=1}^{\infty} \mathrm{P}(N_{2i}) = \sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \frac{1}{4}\frac{1}{1-1/4} = \frac{1}{3} \end{aligned}$$

and hence

$$\mathrm{P}(S_4|E) = 3\frac{433}{20736} = \frac{433}{6912} \approx 0.063.$$

(ii) As above, using the conditional probability definition

$$\begin{aligned}
\mathrm{P}(N_2|S_4) &= \frac{\mathrm{P}(N_2 \cap S_4)}{\mathrm{P}(S_4)} = \frac{\mathrm{P}(S_4|N_2)\mathrm{P}(N_2)}{\sum\limits_{i=1}^{\infty}\mathrm{P}(S_4|N_i)\mathrm{P}(N_i)} \\
&= \frac{\mathrm{P}(S_4|N_2)\mathrm{P}(N_2)}{\mathrm{P}(S_4|N_1)\mathrm{P}(N_1) + \mathrm{P}(S_4|N_2)\mathrm{P}(N_2) + \mathrm{P}(S_4|N_3)\mathrm{P}(N_3) + \mathrm{P}(S_4|N_4)\mathrm{P}(N_4)} \\
&= \frac{\frac{3}{6^2}\frac{1}{4}}{\frac{1}{6}\frac{1}{2} + \frac{3}{6^2}\frac{1}{4} + \frac{3}{6^3}\frac{1}{8} + \frac{1}{6^4}\frac{1}{16}} = \frac{432}{2197} \approx 0.197.
\end{aligned}$$

3. Let W be the event that player 1 wins the game overall, and let W_i be the event that player wins overall on game i. Then

$$\mathrm{P}(W) = \mathrm{P}\left(\bigcup_{i=3}^{\infty} W_i\right) = \mathrm{P}(W_3) + \mathrm{P}(W_4) + \sum_{i=5}^{\infty}\mathrm{P}(W_i).$$

Now

$$\mathrm{P}(W_3) = p^3 \qquad \mathrm{P}(W_4) = \binom{3}{2}p^3(1-p) = 3p^3(1-p) \qquad \mathrm{P}(W_i) = 0 \text{ for } i = 5, 7, 9, ...$$

and

$$P(W_{2i+2}) = \binom{4}{2}p^2(1-p)^2 \times (2p(1-p))^{i-2} \times p^2 = 6p^4(1-p)^2 \times (2p(1-p))^{i-2} \qquad i = 2, 3, 4, ...$$

as W_{2i+2} corresponds to a sequence of games in which the first four end two games all (with probability $6p^2(1-p)^2$), then a sequence of $2(i-2)$ games that ends at $i-2$ games all with no overall winner, that is, $i-2$ pairs of games with one win each in either order (with probability $(2p(1-p))^{i-2}$), then finally two wins for player 1 (with probability p^2). Hence

$$\begin{aligned}
\mathrm{P}(W) &= p^3 + 3p^3(1-p) + 6p^4(1-p)^2\sum_{i=2}^{\infty}(2p(1-p))^{i-2} \\
&= p^3 + 3p^3(1-p) + 6p^4(1-p)^2\sum_{i=0}^{\infty}(2p(1-p))^{i} \\
&= p^3 + 3p^3(1-p) + \frac{6p^4(1-p)^2}{1-2p(1-p)} = \frac{p^3(4-5p+2p^2)}{1-2p+2p^2}
\end{aligned}$$

Easy to show that $g(p) + g(1-p) = 1$, which is reasonable as $g(1-p) = \mathrm{P}(W')$.

4. Let F_n be the event that the weather is fine on day n. Then conditioning on the weather on day $n-1$, and using the Theorem of Total Probability

$$\mathrm{P}(F_n) = \mathrm{P}(F_n|F_{n-1})\mathrm{P}(F_{n-1}) + \mathrm{P}(F_n|F_{n-1}^{'})\mathrm{P}(F_{n-1}^{'}) \quad \therefore \quad \theta_n = p\theta_{n-1} + (1-p)(1-\theta_{n-1})$$

and hence

$$\left(\theta_n - \frac{1}{2}\right) = (2p-1)\left(\theta_{n-1} - \frac{1}{2}\right) = (2p-1)^{n-1}\left(\theta_1 - \frac{1}{2}\right) \quad \therefore \quad \theta_n = \frac{1}{2} + (2p-1)^{n-1}\left(\theta - \frac{1}{2}\right) \longrightarrow \frac{1}{2}$$

as $n \longrightarrow \infty$.

5. Let E and F be the events that the sequence of tosses results in n Heads, and that the coin is fair respectively. Then

$$\mathrm{P}(F|E) = \frac{\mathrm{P}(E|F)\mathrm{P}(F)}{\mathrm{P}(E|F)\mathrm{P}(F) + \mathrm{P}(E|F^{'})\mathrm{P}(F^{'})}$$

(i) $\mathrm{P}(E|F) = \left(\frac{1}{2}\right)^n$, $\mathrm{P}(E|F^{'}) = 1$, $\mathrm{P}(F) = \mathrm{P}(F^{'}) = \frac{1}{2}$, and hence $\mathrm{P}(F|E) = \dfrac{1}{1+2^n}$.

(ii) $\mathrm{P}(E|F) = \binom{n}{k}\left(\frac{1}{2}\right)^n$, $\mathrm{P}(E|F^{'}) = \binom{n}{k}p^k(1-p)^{n-k}$, $\mathrm{P}(F) = \mathrm{P}(F^{'}) = \frac{1}{2}$, and hence

$$\mathrm{P}(F|E) = \frac{1}{1 + 2^n p^k (1-p)^{n-k}}.$$

6. Let E, F and G be the events that the flower produces ripe fruit, that the flower is pollinated, and that the fruit ripens respectively. Then $\mathrm{P}(E) = \mathrm{P}(F \cap G) = \mathrm{P}(F)\mathrm{P}(G) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$.

Now let A_n be the event that the tree produces n flowers, and B_r be the event that the tree produces r ripe fruit (for $n \geq r$). Then

$$\mathrm{P}(A_n|B_r) = \frac{\mathrm{P}(B_r|A_n)\mathrm{P}(A_n)}{\displaystyle\sum_{n=r}^{\infty}\mathrm{P}(B_r|A_n)\mathrm{P}(A_n)}$$

Now

$$\mathrm{P}(B_r|A_n) = \binom{n}{r}\left(\frac{1}{2}\right)^n \qquad \mathrm{P}(A_n) = (1-p)p^n$$

so

$$\begin{aligned}
\mathrm{P}(A_n|B_r) &= \frac{\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n}{\displaystyle\sum_{n=r}^{\infty}\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n} \\
&= \binom{n}{r}\left(\frac{p}{2}\right)^n \frac{1}{\displaystyle\sum_{x=0}^{\infty}\binom{x+r}{r}\left(\frac{p}{2}\right)^{x+r}} = \binom{n}{r}\left(\frac{p}{2}\right)^{n-r} \frac{1}{\left(1-\frac{p}{2}\right)^{-(r+1)}}
\end{aligned}$$

using the binomial expansion for negative exponent. Hence

$$\mathrm{P}(A_n|B_r) = \binom{n}{r}\frac{p^{n-r}(2-p)^{r+1}}{2^{n+1}} \qquad r \leq n$$

7. Let T_k be the event that there are k successive positive tests, let S be the event that drugs are present. Then

$$\mathrm{P}(S|T_k) = \frac{\mathrm{P}(T_k|S)\mathrm{P}(S)}{\mathrm{P}(T_k|S)\mathrm{P}(S) + \mathrm{P}(T_k|S')\mathrm{P}(S')} = \frac{0.99^k \times 0.0002}{0.99^k \times 0.0002 + (1-0.98)^k \times (1-0.0002)}$$

as, by conditional independence

$$\mathrm{P}(T_k|S) = \{\mathrm{P}(T_1|S)\}^k \qquad \mathrm{P}(T_k|S') = \left\{\mathrm{P}(T_1|S')\right\}^k$$

If $k = 1$, $\mathrm{P}(S|T_1) = 0.0098$.

If $k = 2$, $\mathrm{P}(S|T_2) = 0.3289$.