M2S1 : ASSESSED COURSEWORK 2 : SOLUTIONS

(a) Joint density

$$f_{X,Y}(x,y) = c_1 (x+2y) \exp\{-(x+y)\} \qquad x,y>0$$

(i) To compute c_1: need to integrate $f_{X,Y}$ over $\mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{aligned}
\int_0^\infty \int_0^\infty f_{X,Y}(x,y)\,dxdy &= c_1 \int_0^\infty \int_0^\infty (x+2y)\,e^{-(x+y)}dxdy = c_1 \int_0^\infty \left\{\int_0^\infty (x+2y)\,e^{-(x+y)}dx\right\} e^{-y}dy \\
&= c_1 \int_0^\infty \left\{\left[-(x+2y)\,e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}dx\right\} e^{-y}dy \\
&= c_1 \int_0^\infty \left\{2y + \left[e^{-x}\right]_0^\infty\right\} e^{-y}dy \\
&= c_1 \int_0^\infty (2y+1)\,e^{-y}dy \\
&= c_1 \left\{\left[-(2y+1)\,e^{-y}\right]_0^\infty + \int_0^\infty e^{-y}dy\right\} \\
&= c_1\{2+1\} = 3c_1
\end{aligned}$$

and therefore $c_1 = 1/3$.

[2 MARKS]

(ii) For $x>0$

$$\begin{aligned}
f_X(x) &= c_1 \int_0^\infty f_{X,Y}(x,y)dy = \frac{1}{3}\int_0^\infty (x+2y)\,e^{-(x+y)}dy \\
&= \frac{1}{3}e^{-x}\int_0^\infty (x+2y)\,e^{-y}dy \\
&= \frac{1}{3}e^{-x}\left\{\left[-(x+2y)\,e^{-y}\right]_0^\infty + \int_0^\infty 2e^{-y}dy\right\} = \frac{1}{3}e^{-x}\{x+2\}
\end{aligned}$$

and hence

$$f_X(x) = \frac{(x+2)e^{-x}}{3} \qquad x>0$$

[2 MARKS]

Similarly, for $y>0$

$$\begin{aligned}
f_Y(y) &= c_1 \int_0^\infty f_{X,Y}(x,y)dx = \frac{1}{3}\int_0^\infty (x+2y)\,e^{-(x+y)}dx \\
&= \frac{e^{-y}}{3}\left\{\left[-(x+2y)\,e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}dx\right\} = \frac{1}{3}e^{-y}\{2y+1\}
\end{aligned}$$

and hence

$$f_Y(y) = \frac{(2y+1)e^{-y}}{3} \qquad y>0$$

[2 MARKS]

(iv)

$$P\left[Y > X\right] = \int\int_A f_{X,Y}(x,y)dxdy \qquad A \equiv \{(x,y) : 0 < x < y < \infty\}$$

so therefore

$$\begin{aligned}
P\left[Y > X\right] &= \int_0^\infty \int_0^y f_{X,Y}(x,y)dxdy \\
&= \int_0^\infty \left\{ \int_0^y \frac{1}{3}\left(x+2y\right)e^{-(x+y)}dx \right\} dy \\
&= \frac{1}{3}\int_0^\infty \left\{ \left[-(x+2y)e^{-x}\right]_0^y + \int_0^y e^{-x} \right\} e^{-y}dy \\
&= \frac{1}{3}\int_0^\infty \left\{ 2y - 3ye^{-y} + \left(1-e^{-y}\right) \right\} e^{-y}dy \\
&= \frac{1}{3}\int_0^\infty \left\{ (2y+1)e^{-y} - (3y+1)e^{-2y} \right\} dy \\
&= \frac{1}{3}\int_0^\infty \left\{ (2y+1)e^{-y} \right\} dy - \int_0^\infty \left\{ (3y+1)e^{-2y} \right\} dy \\
&= \frac{1}{3}\left\{ \left[-(2y+1)e^{-y}\right]_0^\infty + \int_0^\infty 2e^{-y}dy \right\} - \frac{1}{3}\left\{ \left[-\frac{1}{2}(3y+1)e^{-2y}\right]_0^\infty + \int_0^\infty \frac{3}{2}e^{-2y}dy \right\} \\
&= \frac{1}{3}\left\{1+3\right\} - \frac{1}{3}\left\{\frac{1}{2}+\frac{3}{4}\right\} \\
&= 1 - \frac{1}{3}\times\frac{5}{4} \\
&= \frac{7}{12}
\end{aligned}$$

[2 MARKS]

(b)Joint density

$$f_{X,Y,Z}\left(x,y,z\right) = c_2xyz \qquad 0 < x,y,z < 1$$

(i) To compute c_2: need to integrate $f_{X,Y,Z}$ over $(0,1)\times(0,1)\times(0,1)$

$$\begin{aligned}
\int_0^1\int_0^1\int_0^1 f_{X,Y,Z}\left(x,y,z\right)dxdydz &= c_2\int_0^1\int_0^1\int_0^1 xyz \quad dxdydz \\
&= c_2\left\{\int_0^1 xdx\right\}\times\left\{\int_0^1 ydy\right\}\times\left\{\int_0^1 zdz\right\} \qquad \text{integral factorizes} \\
&= c_2\left\{\left[\frac{x^2}{2}\right]_0^1\right\}\times\left\{\left[\frac{y^2}{2}\right]_0^1\right\}\times\left\{\left[\frac{z^2}{2}\right]_0^1\right\} \\
&= c_2\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}
\end{aligned}$$

and therefore $c_2 = 8$.

Note that we can deduce

$$f_{X,Y,Z}\left(x,y,z\right) = f_X(x)f_Y(y)f_Z(z)$$

where

$$\begin{aligned} f_X(x) &= 2x \qquad 0 < x < 1 \\ f_Y(y) &= 2y \qquad 0 < y < 1 \\ f_Z(z) &= 2z \qquad 0 < z < 1 \end{aligned}$$

and hence X, Y, and Z are **independent**, and identically distributed.

(ii)

$$P\left[X > Y\right] = \int\int_A\int f_{X,Y,Z}(x,y,z)dxdydz \qquad\qquad A \equiv \{(x,y,z) : 0 < x,y,z < 1, x > y\}$$

so therefore

$$\begin{aligned} P\left[X > Y\right] &= \int_0^1 \left\{\int_0^1 \left\{\int_0^x f_{X,Y,Z}\left(x,y,z\right)dy\right\} dx\right\} dz = \int_0^1 \left\{\int_0^1 \left\{\int_0^x 8xyzdy\right\} dx\right\} dz \\ &= 8\int_0^1 \left\{\int_0^1 \left\{\int_0^x ydy\right\} xdx\right\} zdz \\ &= 8\int_0^1 \left\{\int_0^1 \frac{x^2}{2}xdx\right\} zdz = 8\int_0^1 \left\{\int_0^1 \frac{x^3}{2}dx\right\} zdz = 8\int_0^1 \frac{1}{8}zdz = \frac{1}{2} \end{aligned}$$

[3 MARKS]

(iii) By symmetry of the form of the joint density, we can immediately state that

$$P\left[Y > Z\right] = \frac{1}{2}$$

but from first principles,

$$P\left[Y > Z\right] = \int\int_A\int f_{X,Y,Z}(x,y,z)dxdydz \qquad\qquad A \equiv \{(x,y,z) : 0 < x,y,z < 1, y > z\}$$

so

$$\begin{aligned} P\left[Y > Z\right] &= \int_0^1 \left\{\int_0^1 \left\{\int_0^x f_{X,Y,Z}\left(x,y,z\right)dz\right\} dy\right\} dx = \int_0^1 \left\{\int_0^1 \left\{\int_0^y 8xyzdz\right\} dy\right\} dx \\ &= 8\int_0^1 \left\{\int_0^1 \left\{\int_0^y zdz\right\} ydy\right\} xdx \\ &= 8\int_0^1 \left\{\int_0^1 \frac{y^2}{2}ydy\right\} xdx = 8\int_0^1 \left\{\int_0^1 \frac{y^3}{2}dy\right\} xdx = 8\int_0^1 \frac{1}{8}xdx = \frac{1}{2} \end{aligned}$$

[1 MARK]

(c)Joint density

$$f_{X,Y}(x,y) = c_3 \exp\{-2x - y\} \qquad 0 < x < y < \infty$$

(i)For $x > 0$

$$\begin{aligned} f_X(x) &= c_3 \int_0^\infty f_{X,Y}(x,y)\, dy = c_3 \int_x^\infty e^{-(2x+y)} dy \\ &= c_3 e^{-2x} \int_x^\infty e^{-y} dy = c_3 e^{-2x} \left[-e^{-x}\right]_x^\infty = c_3 e^{-3x} \end{aligned}$$

But

$$\int_0^\infty c_3 e^{-3x} dx = c_3 \left[-\frac{e^{-3x}}{3}\right]_0^\infty = \frac{c_3}{3} \qquad \therefore \qquad c_3 = 3$$

and hence

$$f_X(x) = 3e^{-3x} \qquad x > 0$$

so that $X \sim Exponential(3)$.

[2 MARKS]

Similarly for $y > 0$

$$\begin{aligned} f_Y(y) &= c_3 \int_0^\infty f_{X,Y}(x,y)\, dx = c_3 \int_0^y e^{-(2x+y)} dx \\ &= c_3 e^{-y} \int_0^y e^{-2x} dx \\ &= c_3 e^{-y} \left[-\frac{e^{-2x}}{2}\right]_0^y = \frac{c_3}{2} e^{-y}\left(1 - e^{-2y}\right) \end{aligned}$$

and hence

$$f_Y(y) = \frac{3}{2} e^{-y}\left(1 - e^{-2y}\right) \qquad y > 0$$

[2 MARKS]

(ii) X and Y are **not independent**

[1 MARK]

This can be deduced in many ways; for example, if we take a particular point $(x,y) \in \mathbb{R}^2$ such that $x > y$. At this point, $f_{X,Y}(x,y) = 0$, but $f_X(x) > 0$ and $f_Y(y) > 0$ so

$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$$

[1 MARK]