M2S1 : ASSESSED COURSEWORK 1 : SOLUTIONS

(a) We have, for $x = 1, 2, 3, ...9$

$$f_X(x) = k_1 \log_{10}\left(1 + \frac{1}{x}\right)$$

Must check

$$0 \le f_X(x) \le 1 \qquad\qquad \sum_{x=1}^{9} f_X(x) = 1$$

Now, provided $k_1 > 0$,

$$f_X(1) = k_1 \log_{10}\left(1 + \frac{1}{1}\right) = k_1 \log_{10} 2 > 0 \qquad f_X(9) = k_1 \log_{10}\left(1 + \frac{1}{9}\right) > 0$$

and $f_X(x)$ is decreasing in x. Also

$$\begin{aligned}
\log_{10}\left(1 + \frac{1}{x}\right) &= \log_{10}(1+x) - \log_{10}(x) \\
\implies \sum_{x=1}^{9} \log_{10}\left(1 + \frac{1}{x}\right) &= \sum_{x=1}^{9} \left[\log_{10}(1+x) - \log_{10}(x)\right] \\
&= \left[\log_{10}(1+9) - \log_{10}(1)\right] \qquad \text{as the sum telescopes} \\
&= 1
\end{aligned}$$

and hence $k_1 = 1$ and (i) and (ii) hold.

[4 MARKS]

(b) (i) We have from the logarithmic sum given

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z) \implies \sum_{y=1}^{\infty} \frac{\theta^y}{y3^y} = -\log\left(1 - \frac{\theta}{3}\right) \therefore k_2 = -\frac{1}{\log\left(1 - \frac{\theta}{3}\right)}$$

[2 MARKS]

(ii) We have

$$\begin{aligned}
G_Y(t) &= \sum_{y=1}^{\infty} t^y f_Y(y) = \sum_{y=1}^{\infty} t^y \left\{ -\frac{1}{\log\left(1 - \frac{\theta}{3}\right)} \frac{\theta^y}{y3^y} \right\} \\
&= -\frac{1}{\log\left(1 - \frac{\theta}{3}\right)} \left[\sum_{y=1}^{\infty} \frac{1}{y} \left(\frac{\theta t}{3}\right)^y \right] \\
&= -\frac{1}{\log\left(1 - \frac{\theta}{3}\right)} \left[-\log\left(1 - \frac{\theta t}{3}\right) \right] = \frac{\log\left(1 - \frac{\theta t}{3}\right)}{\log\left(1 - \frac{\theta}{3}\right)}
\end{aligned}$$

[4 MARKS]

(c)(i) Conditional on $N = n$

$$E_x = \bigcap_{i=1}^{n} E_{ix} \Longrightarrow P(E_x|N = n) = \prod_{i=1}^{n} P(E_{ix}) = \prod_{i=1}^{n} e^{-\lambda x} = e^{-n\lambda x}$$

as the events $E_{1x}, ..., E_{nx}$ are independent (and thus conditionally independent given $N = n$)

[2 MARKS]

(ii) By the Theorem of Total Probability on the suggested partition,

$$\begin{aligned} P(E_x) &= \sum_{n=0}^{\infty} P(E_x|N = n)P(N = n) \\ &= \sum_{n=0}^{\infty} e^{-n\lambda x} \frac{\mu^n e^{-\mu}}{n!} = e^{-\mu} \sum_{n=0}^{\infty} \frac{\left(\mu e^{-\lambda x}\right)^n}{n!} \\ &= e^{-\mu} \exp\left\{\mu e^{-\lambda x}\right\} \qquad \text{as the sum is of an exponential type} \\ &= \exp\left\{-\mu\left(1 - e^{-\lambda x}\right)\right\} \end{aligned}$$

.

[6 MARKS]

(iii) As $x \to \infty$, $e^{-\lambda x} \to 0$ so

$$P(E_x) \to \exp\{-\mu\} = \theta > 0.$$

Thus there is a positive probability θ that the organism does not succomb; this is due to the fact that

$$P(N = 0) = e^{-\mu} = \theta > 0.$$

[2 MARKS]

M2S1 : SUPPLEMENTARY QUESTIONS 1 : SOLUTIONS

1. Need $\sum_{x=1}^{\infty} f_x(x) = 1$. Hence

$$\text{(a)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{2^x} = 1 \qquad \text{(b)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x2^x} = \log 2$$

$$\text{(c)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \qquad \text{(d)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{2^x}{x!} = e^2 - 1$$

(a) is given by the sum of a geometric progression; (b) uses the fact that

$$\frac{1}{1-t} = 1 + t + t^2 + ... = \sum_{x=0}^{\infty} t^x \Longrightarrow -\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + ... = \sum_{x=1}^{\infty} \frac{t^x}{x}$$

by integrating both sides with respect to t. Hence for $t = 1/2$, we have

$$\log 2 = -\log(1 - 1/2) = \sum_{x=1}^{\infty} \frac{1}{x2^x}.$$

(c) is a well-known mathematical result ...; (d) uses the power series expansion of e^t, evaluated at $t = 2$, that is

$$e^t = \sum_{x=0}^{\infty} \frac{t^x}{x!} \Longrightarrow e^2 = \sum_{x=0}^{\infty} \frac{2^x}{x!} = 1 + \sum_{x=1}^{\infty} \frac{2^x}{x!}$$

Clearly $\mathrm{P}[\ X > 1\ ] = 1 - \mathrm{P}[\ X = 1\ ]$, so

$$\text{(a)} \quad \mathrm{P}[\ X > 1\ ] = \frac{1}{2} \qquad \text{(b)} \quad \mathrm{P}[\ X > 1\ ] = 1 - \frac{1}{2\log 2}$$

$$\text{(c)} \quad \mathrm{P}[\ X > 1\ ] = 1 - \frac{6}{\pi^2} \qquad \text{(d)} \quad \mathrm{P}[\ X > 1\ ] = \frac{e^2 - 3}{e^2 - 1}$$

$\mathrm{P}[X \text{ is even }] = \sum_{x=1}^{\infty} \mathrm{P}[\ X = 2i\ ]$, so

$$\text{(a)} \quad \mathrm{P}[\ X \text{ is even }] = \frac{1}{3} \qquad \text{(b)} \quad \mathrm{P}[\ X \text{ is even }] = 1 - \frac{\log 3}{\log 4}$$

$$\text{(c)} \quad \mathrm{P}[\ X \text{ is even }] = \frac{1}{4} \qquad \text{(d)} \quad \mathrm{P}[\ X \text{ is even }] = \frac{1 - e^{-2}}{2}$$

(a) is still the sum of a geometric progression

(b) follows from the previous result

(c) follows from the previous result taking out a factor of 1/4

(d) uses the sum of the two power series of e^t and e^{-t}, to knock out the odd terms, evaluated at $t = 2$.

2. Let Z and X be the numbers of Heads obtained on the first and second tosses respectively. Then the ranges of Z and X are both $\{0, 1, 2, ..., n\}$. Now

$$f_X(x) = \mathrm{P}[\ X = x\ ] = \sum_{z=1}^{n} \mathrm{P}[\ X = x \mid Z = z\ ]\ \mathrm{P}[\ Z = z\ ] = \sum_{z=x}^{n} \binom{z}{x}\left(\frac{1}{2}\right)^z \binom{n}{z}\left(\frac{1}{2}\right)^n$$

using the Theorem of Total probability. Hence

$$f_X(x) = \left(\frac{1}{2}\right)^n \sum_{z=x}^{n} \frac{z!}{x!(z-x)!}\frac{n!}{z!(n-z)!}\left(\frac{1}{2}\right)^z = \left(\frac{1}{2}\right)^n \binom{n}{x}\sum_{z=x}^{n}\binom{n-x}{n-z}\left(\frac{1}{2}\right)^z$$

But

$$\sum_{z=x}^{n}\binom{n-x}{n-z}\left(\frac{1}{2}\right)^z = \sum_{t=0}^{m}\binom{m}{m-t}\left(\frac{1}{2}\right)^{t+x} = \left(\frac{1}{2}\right)^x\left(1+\frac{1}{2}\right)^m$$

where $t = z - x$, and $m = n - x$, using the Binomial Expansion. Hence

$$f_X(x) = \binom{n}{x}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^x\left(1+\frac{1}{2}\right)^{n-x} = \binom{n}{x}\frac{3^{n-x}}{2^{2n}} \qquad x = 0, 1, 2, ..., n.$$

Alternately, as all tosses are independent, consider tossing all n coins twice, and counting the number that show heads twice; this is identical to evaluating X. Then as each coin shows heads twice with probability $\left(\frac{1}{2}\right)^2$,

$$f_X(x) = \binom{n}{x}\left\{\left(\frac{1}{2}\right)^2\right\}^x\left\{1-\left(\frac{1}{2}\right)^2\right\}^{n-x} = \binom{n}{x}\frac{3^{n-x}}{2^{2n}}$$

as before.

3. Each of the $n(n+1)/2$ points has equal probability $p = 2/(n(n+1))$ of being selected. In column x of the triangular array of points, there are x points in total; in row y, there are $(n+1-y)$ points (for $x, y = 1, 2, ..., n$) and therefore

$$f_X(x) = \mathrm{P}[\ X = x\ ] = xp = \frac{2x}{n(n+1)} \qquad x = 1, 2, ..., n$$

$$f_Y(y) = \mathrm{P}[\ Y = y\ ] = (n+1-y)p = \frac{2(n+1-y)}{n(n+1)} \qquad y = 1, 2, ..., n$$

4. Can calculate F_X by integration

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x ct^2(1-t)\ dt = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right] \qquad 0 < x < 1$$

and $F_X(1) = 1$ gives $c = 12$. Finally, $\mathrm{P}[\ X > 1/2\ ] = 1 - \mathrm{P}[X \leq 1/2] = 1 - F_X(1/2) = 1 - 12[1/24 - 1/64] = 11/16$.

5. Valid pdf if (i) it is a non-negative function (that is, if $k > 0$), and (ii) integrates to 1 over the range $x > 1$, that is

$$\int_1^{\infty} f_X(x)\ dx = \int_1^{\infty} \frac{k}{x^{k+1}}\ dx = \left[-\frac{1}{x^k}\right]_1^{\infty} = 1 \ \text{ if } k > 0$$

so f_X is a pdf if $k > 0$, and $F_X(x) = 1 - \dfrac{1}{x^k}$ for $x > 1$.

6. Sketch of f_X;

$$F_X(x) = \int_{-\infty}^{x} f_x(t)\, dt = \begin{cases} \displaystyle\int_0^x t\, dt & = \dfrac{x^2}{2} & 0 < x < 1 \\ \displaystyle\int_0^1 t\, dt + \int_1^x (2-t)\, dt & = 2x - \dfrac{x^2}{2} - 1 & 1 \leq x < 2 \end{cases}$$

Note that F_X is continuous, and $F_X(0) = 0$, $F_X(2) = 1$.

7. $F_X(1) = 1 \Longrightarrow \dfrac{1}{\alpha - \beta}$, and

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x} = \frac{\alpha\beta}{\alpha-\beta}\left(x^{\beta-1} - x^{\alpha-1}\right) \qquad 0 \leq x \leq 1$$

and zero otherwise, and hence

$$\begin{aligned} \mathrm{E}_{f_X}[\, X^r \,] &= \int_{-\infty}^{\infty} x^r f_X(x)\, dx = \int_0^1 \frac{\alpha\beta}{\alpha-\beta}\left(x^{\beta-1} - x^{\alpha-1}\right)\, dx \\ &= \frac{\alpha\beta}{\alpha-\beta}\left[\frac{x^{\beta+r}}{\beta+r} - \frac{x^{\alpha+r}}{\alpha+r}\right]_0^1 \\ &= \frac{\alpha\beta}{(\alpha+r)(\beta+r)} \end{aligned}$$

8. By differentiation,

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x} = \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2} \qquad 0 \leq x \leq \beta$$

and zero otherwise, and hence

$$\begin{aligned}
\mathrm{E}_{f_X}[\,X\,] &= \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_0^{\beta} x \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2}\ dx \\
&= \int_0^{\pi/4} 2\beta^2 \tan^2\theta \frac{\beta^2(1 - \tan^2\theta)}{\beta^4(1 + \tan^2\theta)^2} \beta \sec^2\theta\ d\theta \qquad (x = \beta\tan\theta) \\
&= 2\beta \int_0^{\pi/4} \tan\theta\ \frac{(1 - \tan^2\theta)}{(1 + \tan^2\theta)}\ d\theta \\
&= 2\beta \int_0^{\pi/4} \tan\theta \cos 2\theta\ d\theta \\
&= 2\beta \left[\frac{1}{2}\tan\theta\sin 2\theta\right]_0^{\pi/4} - \beta \int_0^{\pi/4} \sec^2\theta \sin 2\theta\ d\theta \qquad \text{(by parts)} \\
&= 2\beta \left[\frac{1}{2} - \int_0^{\pi/4} \tan\theta\ d\theta\right] \\
&= 2\beta \left[\frac{1}{2} - \left[-\log(\cos\theta)\right]_0^{\pi/4}\right] \\
&= 2\beta \left[\frac{1}{2} + \log(\cos\pi/4)\right] = \beta(1 - \log 2)
\end{aligned}$$

as $\cos\pi/4 = 1/\sqrt{2}$.

9. Using the formula given in the question for f_S

$$f_S(s) = \sum_{x=0}^{s} f_X(x) f_Y(s-x)$$

we have for the pgf

$$\begin{aligned} G_S(t) &= \sum_{s=0}^{\infty} t^s f_S(s) = \sum_{s=0}^{\infty} t^s \left\{ \sum_{x=0}^{s} f_X(x) f_Y(s-x) \right\} = \sum_{s=0}^{\infty} \sum_{x=0}^{s} t^s f_X(x) f_Y(s-x) \\ &= \sum_{x=0}^{\infty} \sum_{s=x}^{\infty} t^s f_X(x) f_Y(s-x) \\ &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} t^{x+y} f_X(x) f_Y(y) \qquad \text{(setting } y = s-x\text{)} \\ &= \left\{ \sum_{x=0}^{\infty} t^x f_X(x) \right\} \left\{ \sum_{y=0}^{\infty} t^y f_Y(y) \right\} = G_X(t) G_Y(t) \end{aligned}$$

10. (i) For the pmf given (which is Binomial with parameters n and θ)

$$\begin{aligned} G_{S_i}(t) &= \sum_{x=0}^{\infty} t^x f_{S_i}(x) = \sum_{x=0}^{n} t^x \binom{n}{x} \theta^x (1-\theta)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (\theta t)^x (1-\theta)^{n-x} \\ &= (1-\theta+\theta t)^n \end{aligned}$$

by the binomial theorem.

(ii) **Given** that $R = r$, we seek the pmf of the sum

$$S = \sum_{i=1}^{r} S_i$$

which we can compute using the a key pgf result from lectures which is the extension of the result in (i), namely that as the S_i variables are independent we have for the conditional pgf

$$G_S(t) = \prod_{i=1}^{r} G_{S_i}(t)$$

so that from (i),

$$G_S(t) = \prod_{i=1}^{r} (1-\theta+\theta t)^n = (1-\theta+\theta t)^{nr}$$

which is a pgf of a form identical to that in (ii) but with a different power. Hence this is the pgf coresponding to the pmf

$$\binom{nr}{x} \theta^x (1-\theta)^{nr-x} \qquad x \in \{0, 1, ..., nr\}$$

Hence if the conditional pmf given $R = r$ is denoted $f_{S|R}(s|r)$ then

$$f_{S|R}(s|r) = \mathrm{P}[S = s|R = r] = \binom{nr}{s} \theta^s (1-\theta)^{nr-s} \qquad s \in \{0, 1, ..., nr\}$$

(iii) Finally, the unconditional pmf of S, $f_S(s)$, is obtained from (ii) using the Theorem of Total Probability, as

$$\begin{aligned} f_S(s) &= \mathrm{P}[S = s] = \sum_{r=0}^{\infty} \mathrm{P}[S = s|R = r]\,\mathrm{P}[R = r] \\ &= \sum_{r=0}^{\infty} \binom{nr}{s} \theta^s (1-\theta)^{nr-s} \frac{e^{-\lambda}\lambda^r}{r!} \qquad s = 0, 1, 2, ... \end{aligned}$$

which is an expression that cannot be simplified usefully. However, if we try to compute the (unconditional) pgf of S then

$$\begin{aligned} G_S(t) &= \sum_{s=0}^{\infty} t^s f_S(s) = \sum_{s=0}^{\infty} t^s \left\{ \sum_{r=0}^{\infty} \mathrm{P}[S = s|R = r]\,\mathrm{P}[R = r] \right\} \\ &= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} t^s \mathrm{P}[S = s|R = r]\,\mathrm{P}[R = r] \\ &= \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{nr} t^s \mathrm{P}[S = s|R = r] \right\} \mathrm{P}[R = r] && \text{exchanging order of summation, noting } \mathrm{P}[S = s|R = r] = 0 \text{ if } s > nr \\ &= \sum_{r=0}^{\infty} \left\{ \sum_{s=0}^{nr} t^s \binom{nr}{s} \theta^s (1-\theta)^{nr-s} \right\} \mathrm{P}[R = r] \\ &= \sum_{r=0}^{\infty} \{(1-\theta+\theta t)^{nr}\}\,\mathrm{P}[R = r] && \text{from (ii)} \\ &= \sum_{r=0}^{\infty} (1-\theta+\theta t)^{nr} \frac{e^{-\lambda}\lambda^r}{r!} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda(1-\theta+\theta t)^n)^r}{r!} \\ &= e^{-\lambda} \exp\{\lambda(1-\theta+\theta t)^n\} && \text{as the sum is that of an exponential type.} \\ &= \exp\{-\lambda(1-(1-\theta+\theta t)^n\} \end{aligned}$$