CHAPTER 1

DEFINITIONS, TERMINOLOGY, NOTATION

1.1 EVENTS AND THE SAMPLE SPACE

Definition 1.1.1 An **experiment** is a one-off or repeatable process or procedure for which
(a) there is a well-defined set of *possible* outcomes
(b) the *actual* outcome is not known with certainty.

Definition 1.1.2 A **sample outcome**, ω, is precisely one of the possible outcomes of an experiment.

Definition 1.1.3 The **sample space**, Ω, of an experiment is the set of all possible outcomes.

NOTE : Ω is a set in the mathematical sense, so set theory notation can be used. For example, if the sample outcomes are denoted $\omega_1, ..., \omega_k$, say, then

$$\Omega = \{\omega_1, ..., \omega_k\} = \{\omega_i : i = 1, ..., k\},$$

and $\omega_i \in \Omega$ for $i = 1, ..., k$.

The sample space of an experiment can be

- a FINITE list of sample outcomes, $\{\omega_1, ..., \omega_k\}$
- an INFINITE list of sample outcomes, $\{\omega_1, \omega_2, ...\}$
- an INTERVAL or REGION of a real space, $\left\{\omega : \omega \in A \subseteq \mathbb{R}^d\right\}$

Definition 1.1.4 An **event**, E, is a designated collection of sample outcomes. Event E **occurs** if the actual outcome of the experiment is one of this collection.

Special Cases of Events

The event corresponding to collection of *all* sample outcomes is Ω.

The event corresponding to a collection of *none* of the sample outcomes is denoted $\emptyset$.

i.e. The sets $\emptyset$ and Ω are also events, termed the **impossible** and the **certain** event respectively, and for any event E, $E \subseteq \Omega$.

1.1.1 OPERATIONS IN SET THEORY

Set theory operations can be used to manipulate events in probability theory. Consider events $E, F \subseteq \Omega$. Then the three basic operations are

UNION	$E \cup F$	"E or F or both occur"
INTERSECTION	$E \cap F$	"both E and F occur"
COMPLEMENT	E'	"E does not occur"

Properties of Union/Intersection operators

Consider events $E, F, G \subseteq \Omega$.

COMMUTATIVITY	$E \cup F = F \cup E$ $E \cap F = F \cap E$
ASSOCIATIVITY	$E \cup (F \cup G) = (E \cup F) \cup G$ $E \cap (F \cap G) = (E \cap F) \cap G$
DISTRIBUTIVITY	$E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$ $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$
DE MORGAN'S LAWS	$(E \cup F)^{'} = E^{'} \cap F^{'}$ $(E \cap F)^{'} = E^{'} \cup F^{'}$

NOTE: Union and intersection are *binary* operators, that is, they take only two arguments, and thus the bracketing in the above equations is necessary. For $k \geq 2$ events, $E_1, E_2, ..., E_k$,

$$\bigcup_{i=1}^{k} E_i = E_1 \cup ... \cup E_k \qquad \text{and} \qquad \bigcap_{i=1}^{k} E_i = E_1 \cap ... \cap E_k$$

for the union and intersection of $E_1, E_2, ..., E_k$. with a further extension for k *infinite.*

1.1.2 MUTUALLY EXCLUSIVE EVENTS AND PARTITIONS

Definition 1.1.5 Events E and F are **mutually exclusive** if $E \cap F = \emptyset$, that is, if events E and F cannot both occur. If the sets of sample outcomes represented by E and F are **disjoint** (have no common element), then E and F are mutually exclusive.

Definition 1.1.6 Events $E_1, ..., E_k \subseteq \Omega$ form a **partition** of event $F \subseteq \Omega$ if
(a) $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, ..., k$
(b) $\bigcup_{i=1}^{k} E_i = F$.

so that each element of the collection of sample outcomes corresponding to event F is in *one and only one* of the collections corresponding to events $E_1, ...E_k$.

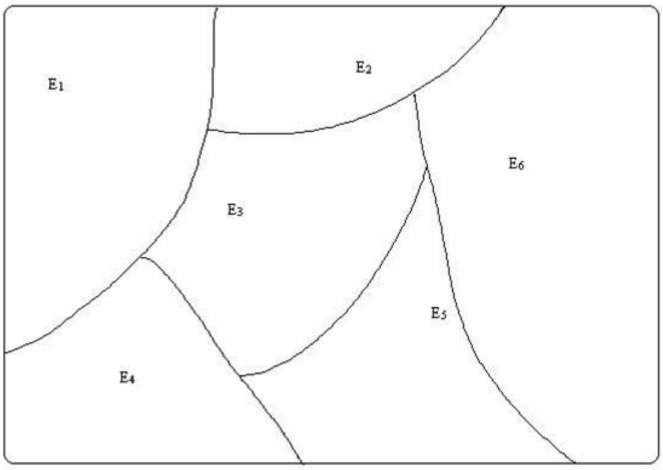

Figure 1.1: Partition of Ω

In Figure 1.1, we have

$$\Omega = \bigcup_{i=1}^{6} E_i$$

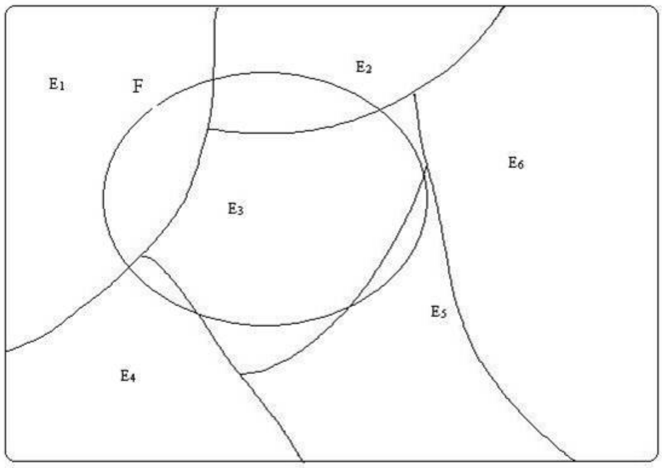

Figure 1.2: Partition of $F \subset \Omega$

In Figure 1.2, we have

$$F = \bigcup_{i=1}^{6} (F \cap E_i)\,,\text{ but, for example, } F \cap E_6 = \emptyset$$

Construction of disjoint events from general events

Suppose that $A_1, A_2, ..., A_k, ...$ is a (countable) collection of general events in sample space Ω. Then the collection of events $E_1, E_2, ..., E_k, ...$ defined for $k \geq 1$ by

$$E_k = A_k \bigcap \left(\bigcup_{i=1}^{k-1} A_i \right)'$$

are pairwise mutually exclusive, but

$$\bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} A_i.$$

E_k is the set of sample outcomes in Ω that are elements of A_k but not elements of any of $A_1, ..., A_{k-1}$.

1.1.3 ALGEBRAS AND SIGMA ALGEBRAS*

Definition 1.1.7 A (countable) collection of subsets, $\mathcal{A}$, of sample space Ω, say

$$\mathcal{A} = \{A_1, A_2, ...\},$$

is an **algebra** if
(I)$\Omega \in \mathcal{A}$
(II)$A_1, A_2 \in \mathcal{A} \Longrightarrow A_1 \cup A_2 \in \mathcal{A}$
(III)$A \in \mathcal{A} \Longrightarrow A' \in \mathcal{A}$

Interpretation: An algebra is a set of sets (events) with certain properties, in particular it is *closed* under a **finite** number of union operations (II), that is if $A_1, ...A_k \in \mathcal{A}$, then

$$\bigcup_{i=1}^{k} A_i \in \mathcal{A}.$$

NOTE: If $\mathcal{A}$ is an algebra of subsets of Ω, then

(i) $\emptyset \in \mathcal{A}$

(ii) If $A_1, A_2 \in \mathcal{A}$, then

$$A_1', A_2' \in \mathcal{A} \quad \Longrightarrow \quad A_1' \cup A_2' \in \mathcal{A} \quad \Longrightarrow \quad \left(A_1' \cup A_2'\right)' \in \mathcal{A} \quad \Longrightarrow \quad A_1 \cap A_2 \in \mathcal{A}$$

so $\mathcal{A}$ is also *closed under intersection.*

Extension: A **sigma-algebra ($\sigma-$algebra)** is an algebra that is closed under **countable** union, that is, if $A_1, ...A_k, ... \in \mathcal{A}$, then

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}.$$

1.1.4 SEQUENCES OF EVENTS AND THEIR LIMITS*

Definition 1.1.8 A **sequence** of events $A_1, A_2, ..., A_k, ...$, denoted $\{A_k\}$, is **non-increasing** if

$$A_1 \supseteq A_2 \supseteq ... \supseteq A_k \supseteq ...$$

and is **non-decreasing** if

$$A_1 \subseteq A_2 \subseteq ... \subseteq A_k \subseteq ...$$

A sequence that is either non-increasing or non-decreasing is termed **monotone**.

NOTES:

(i) If $\{A_k\}$ is non-increasing, then the sequence $\{A'_k\}$ is non-decreasing, as for $k \geq 1$,

$$A_k \supseteq A_{k+1} \Longrightarrow A_k = A_{k+1} \cup D_k \text{for some event} D_k$$

$$\Longrightarrow A'_k = (A_{k+1} \cup D_k)\prime = A'_{k+1} \cap D'_k \subseteq A'_{k+1}$$

Hence, for $k \geq 1$, $A'_k \subseteq A'_{k+1}$.

(ii) If $\{A_k\}$ is non-decreasing, then $\{A'_k\}$ is non-increasing. Also, for $k \geq 1$, define event E_k by

$$E_k = A_{k+1} \cap A'_k.$$

Then $\{E_k\}$ is a collection of **disjoint** events, and

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$$

Definition 1.1.9 Consider an infinite monotone sequence of events $\{A_k\}$. Then the **limit event** of the sequence, A, is written

$$A = \lim_{k \longrightarrow \infty} A_k$$

and is defined by

$$A = \begin{cases} \bigcap\limits_{k=1}^{\infty} A_k & \text{if} \{A_k\} \text{isNON-INCREASING} \\ \\ \bigcup\limits_{k=1}^{\infty} A_k & \text{if} \{A_k\} \text{isNON-DECREASING} \end{cases}$$

Interpretation: If $\{A_k\}$ is non-increasing, then the limit event A corresponds to the collection of sample outcomes that are elements of **all** of the events in the sequence. Thus A occurs if and only if all of $A_1, A_2, ..., A_k, ...$ occur.

If $\{A_k\}$ is non-decreasing, then the limit event A corresponds to the collection of sample outcomes that are elements of **at least one** of the events in the sequence. Thus A occurs if and only if at least one of $A_1, A_2, ..., A_k, ...$ occurs.

NOTE: These limit definitions are only valid for monotone sequences of events.

1.2 THE PROBABILITY FUNCTION

Definition 1.2.1 For an event $E \subseteq \Omega$, the **probability that** E **occurs** is written $\mathrm{P}(E)$.

[*Formally, we define probability triple* $(\Omega, \mathcal{A}, P)$ *where* Ω *is the sample space,* $\mathcal{A}$ *is a* σ*-algebra of subsets of* Ω*, and* P *is a probability "measure" operating on* $\mathcal{A}$*.*]

Interpretation : P(.) is a *set-function* that assigns "weight" to collections of possible outcomes of an experiment. There are many ways to think about precisely how this assignment is achieved;

CLASSICAL : "Consider equally likely sample outcomes ..."

FREQUENTIST : "Consider long-run *relative frequencies* ..."

SUBJECTIVE : "Consider personal degree of belief ..."

or merely think of P(.) as a set-function.

1.3 PROPERTIES OF P(.): THE AXIOMS OF PROBABILITY

Consider sample space Ω. Then probability function P(.) satisfies the following properties:

<u>AXIOM 1</u> Let $E \subseteq \Omega$. Then $0 \leq \mathrm{P}(E) \leq 1$.

<u>AXIOM 2</u> $\mathrm{P}(\Omega) = 1$.

<u>AXIOM 3</u> If $E, F \subseteq \Omega$, with $E \cap F = \emptyset$, then $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F)$.

NOTE : Axiom 3 can be re-stated if we can consider an algebra $\mathcal{A}$ of subsets of Ω. If events $A_1, A_2, ...$ are disjoint elements of $\mathcal{A}$, then replace Axiom 3 by requiring that, for $n \geq 1$,

<u>AXIOM 3*</u> $$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{P}(A_i).$$

Furthermore, if $\mathcal{A}$ is a σ-algebra, then Axiom 3* can be replaced by requiring that

<u>AXIOM 3†</u> $$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i).$$

Note that, if $\mathcal{A}$ is a σ-algebra, then

$$\text{AXIOM } 3^{\dagger} \quad \Longrightarrow \quad \text{AXIOM } 3^{*} \quad \Longrightarrow \quad \text{AXIOM } 3$$

that is,

$$\text{COUNTABLE ADDITIVITY} \quad \Longrightarrow \quad \text{FINITE ADDITIVITY} \quad \Longrightarrow \quad \text{ADDITIVITY}$$

1.3.1 COROLLARIES

For events $E, F \subseteq \Omega$

P(E') = 1−P(E), and hence P($\emptyset$) = 0.

If $E \subseteq F$, then P(E) ≤P(F).

In general, P($E \cup F$) =P(E)+P(F)−P($E \cap F$).

P($E \cap F'$) =P(E)−P($E \cap F$)

Boole's Inequality

$$P(E \cup F) \leq P(E) + P(F).$$

Bonferroni's Inequality

$$P(E \cap F) \geq P(E) + P(F) - 1.$$

NOTE: The **general addition rule** for probabilities and Boole's Inequality extend to more than two events. Let $E_1, ..., E_n$ be events in Ω. Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - ... + (-1)^n P\left(\bigcap_{i=1}^{n} E_i\right)$$

and

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i).$$

To prove these results, construct the events $F_1 = E_1$ and

$$F_i = E_i \cap \left(\bigcup_{k=1}^{i-1} E_k\right)'$$

for $i = 2, 3, ..., n$.

Then $F_1, F_2, ...F_n$ are disjoint, and $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, so

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} P(F_i).$$

Now, by Corollary five above

$$\begin{aligned} P(F_i) &= P(E_i) - P\left(E_i \cap \bigcup_{k=1}^{i-1} E_k\right) \qquad i = 2, 3, ..., n. \\ &= P(E_i) - P\left(\bigcup_{k=1}^{i-1} (E_i \cap E_k)\right) \end{aligned}$$

and the result follows by recursive expansion of the second term for $i = 2, 3, ...n$.

1.3.2 PROBABILITY IS A CONTINUOUS SET FUNCTION*

Suppose Axiom $3^\dagger$ holds for a given Ω, and let $A_1, A_2, ... \subseteq \Omega$ be a non-decreasing sequence of events. Then, if A is the limit event for the monotone sequence, we have

$$\begin{aligned} P(A) &= \mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathrm{P}\left(A_1 \cup \left(\bigcup_{i=2}^{\infty}(A_i \cap A'_{i-1})\right)\right) \\ &= \mathrm{P}(A_1) + \sum_{i=2}^{\infty} P(A_i \cap A'_{i-1}) \qquad \text{by Axiom } 3^\dagger \\ &= P(A_1) + \lim_{n\to\infty}\left\{\sum_{i=2}^{n}\left[P(A_i) - P(A_{i-1})\right]\right\} = \lim_{n\to\infty} P(A_n) \end{aligned}$$

as the sequence A_1, $A_2 \cap A'_1$, $A_3 \cap A'_2$, $A_4 \cap A'_3$ and so on is a disjoint sequence of events whose union is identical to A.

Hence P is a *continuous* function as

$$P(A) = P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

with similar result holds for non-increasing sequences. Note also that the converse of this result is also true, that is, if P is continuous, then Axiom $3^\dagger$ holds, and P is countably additive.

1.4 CONDITIONAL PROBABILITY

Definition 1.4.1 For events $E, F \subseteq \Omega$ the **conditional probability** that F occurs **given** that E occurs is written $\mathrm{P}(F|E)$, and is defined by

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

if $\mathrm{P}(E) > 0$.

NOTE: $\mathrm{P}(E \cap F) = \mathrm{P}(E)\mathrm{P}(F|E)$, and in general, for events $E_1, ..., E_k$,

$$P\left(\bigcap_{i=1}^{k} E_i\right) = P(E_1)P(E_2|E_1)P(E_2|E_1 \cap E_2)...P(E_k|E_1 \cap E_2 \cap ... \cap E_{k-1}).$$

This result is known as the CHAIN or MULTIPLICATION RULE.

Definition 1.4.2 Events E and F are **independent** if

$$P(E|F) = P(E) \text{ so that } P(E \cap F) = P(E)P(F)$$

Extension : Events $E_1, ..., E_k$ are independent if, for **every** subset of events of size $l \leq k$, indexed by $\{i_1, ..., i_l\}$, say,

$$P\left(\bigcap_{j=1}^{l} E_{i_j}\right) = \prod_{j=1}^{l} P(E_{i_j}).$$

1.5 THE THEOREM OF TOTAL PROBABILITY

THEOREM

Let $E_1, ..., E_k$ be a partition of Ω, and let $F \subseteq \Omega$. Then

$$P(F) = \sum_{i=1}^{k} P(F|E_i)P(E_i)$$

Proof

$E_1, ..., E_k$ form a partition of Ω, and $F \subseteq \Omega$, so

$$\begin{aligned} F &= (F \cap E_1) \cup ... \cup (F \cap E_k) \\ \implies \quad P(F) &= \sum_{i=1}^{k} P(F \cap E_i) = \sum_{i=1}^{k} P(F|E_i)P(E_i) \end{aligned}$$

(by AXIOM 3*, as$E_i \cap E_j = \emptyset$).

Extension: If we assume that Axiom $3^\dagger$ holds, that is, that P is countably additive, then the theorem still holds, that is, if $E_1, E_2, ...$ are a partition of Ω, and $F \subseteq \Omega$, then

$$P(F) = \sum_{i=1}^{\infty} P(F \cap E_i) = \sum_{i=1}^{\infty} P(F|E_i)P(E_i)$$

if P$(E_i) > 0$ for all i.

1.6 BAYES THEOREM

THEOREM

Suppose $E, F \subseteq \Omega$, with P(E),P$(F) > 0$. Then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

Proof

$$P(E|F)P(F) = P(E \cap F) = P(F|E)P(E), \text{ so} P(E|F)P(F) = P(F|E)P(E).$$

Extension: If $E_1, ..., E_k$ are disjoint, with P$(E_i) > 0$ for $i = 1, ..., k$, and form a partition of $F \subseteq \Omega$, then

$$P(E_i|F) = \frac{P(F|E_i)P(E_i)}{\displaystyle\sum_{i=1}^{k} P(F|E_i)P(E_i)}$$

The extension to the countably additive (infinite) case also holds.

NOTE: in general, P$(E|F) \neq$P$(F|E)$ - see Medical Testing examples.

1.7 COUNTING TECHNIQUES

Suppose that an experiment has N equally likely sample outcomes. If event E corresponds to a collection of sample outcomes of size $n(E)$, then

$$P(E) = \frac{n(E)}{N}$$

so it is necessary to be able to evaluate $n(E)$ and N in practice.

Multiplication principle

If operations labelled $1, ..., r$ can be carried out in $n_1, ..., n_r$ ways respectively, then there are

$$\prod_{i=1}^{r} n_i = n_1 ... n_r$$

ways of carrying out the r operations in total.

EXAMPLE : If each of r trials of an experiment has N possible outcomes, then there are N^r possible sequences of outcomes in total.

e.g.(i) If a multiple choice exam has 20 questions, each of which has 5 possible answers, then there are 5^{20} different ways of completing the exam.

(ii) There are 2^m subsets of m elements (as each element is either **in** the subset, or **not in** the subset, which is equivalent to m trials each with two outcomes).

Sampling from a finite population

Consider a collection of N items, and a sequence of operations labelled $1, ..., r$ such that the ith operation involves **selecting** one of the items remaining after the first $i-1$ operations have been carried out. Let n_i denote the number of ways of carrying out the ith operation, for $i = 1, ..., r$. Then there are two distinct cases;

(a) **Sampling with replacement :** an item is returned to the collection after selection. Then $n_i = N$ for all i, and there are N^r ways of carrying out the r operations.

(b) **Sampling without replacement :** an item is not returned to the collection after selected. Then $n_i = N - i + 1$, and there are $N(N-1)...(N-r+1)$ ways of carrying out the r operations.

e.g. Consider selecting 5 cards from 52. Then

(a) leads to 52^5 possible selections, whereas

(b) leads to 52.51.50.49.48 possible selections

Note: The **order** in which the operations are carried out may be important

e.g. in a raffle with three prizes and 100 tickets, the draw $\{45, 19, 76\}$ is different from $\{19, 76, 45\}$.

Note: the items may be **distinct** (unique in the collection), or **indistinct** (of a unique type in the collection, but not unique individually).

e.g. The numbered balls in the National Lottery, or individual playing cards, are **distinct**. However balls in the lottery are regarded as “WINNING” or “NOT WINNING”, or playing cards are regarded in terms of their suit only, are **indistinct**.

PERMUTATIONS AND COMBINATIONS

Definition 1.7.1 A **permutation** is an *ordered* arrangement of a set of items.
A **combination** is an *unordered* arrangement of a set of items.

RESULT 1 The number of permutations of n distinct items is $n! = n(n-1)...1$.

RESULT 2 The number of permutations of r from n distinct items is

$$P_r^n = \frac{n!}{(n-r)!} = n(n-1)...(n-r+1) \qquad \text{(by the Multiplication Principle)}.$$

If the **order** in which items are selected is not important, then

RESULT 3 The number of combinations of r from n distinct items is

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (\text{as} P_r^n = r!C_r^n).$$

-recall the **Binomial Theorem**, namely

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Then the number of subsets of m items can be calculated as follows; for each $0 \leq j \leq m$, choose a subset of j items from m. Then

$$\text{Total number of subsets} = \sum_{j=0}^{m} \binom{m}{j} = (1+1)^m = 2^m.$$

If the items are **indistinct**, but each is of a unique type, say Type I, ..., Type κ say, (the so-called **Urn Model**) then

RESULT 4 The number of distinguishable permutations of n indistinct objects, comprising n_i items of type i for $i = 1, ..., \kappa$ is

$$\frac{n!}{n_1!n_2!...n_\kappa!}$$

Special Case : if $\kappa = 2$, then the number of distinguishable permutations of the n_1 objects of type I, and $n_2 = n - n_1$ objects of type II is

$$C_{n_2}^n = \frac{n!}{n_1!(n-n_1)!}$$

Also, there are C_r^n ways of partitioning n **distinct** items into two "cells", with r in one cell and $n - r$ in the other.

PROBABILITY CALCULATIONS

Recall that if an experiment has N equally likely sample outcomes, and event E corresponds to a collection of sample outcomes of size $n(E)$, then

$$P(E) = \frac{n(E)}{N}$$

EXAMPLE 1. A True/False exam has 20 questions. Let E = "16 answers correct at random". Then

$$P(E) = \frac{\text{Number of ways of getting 16 out of 20 correct}}{\text{Total number of ways of answering 20 questions}} = \frac{\binom{20}{16}}{2^{20}} = 0.0046$$

EXAMPLE 2. *Sampling without replacement.*

Consider an Urn Model with 10 Type I objects and 20 Type II objects, and an experiment involving sampling five objects without replacement. Let E="precisely 2 Type I objects selected" We need to calculate N and $n(E)$ in order to calculate P(E). In this case N is the number of ways of choosing 5 from 30 items, and hence

$$N = \binom{30}{5}$$

To calculate $n(E)$, we think of E occurring by first choosing 2 Type I objects from 10, and then choosing 3 Type II objects from 20, and hence, by the multiplication rule,

$$n(E) = .\binom{10}{2}\binom{20}{3}$$

Therefore

$$P(E) = \frac{.\binom{10}{2}\binom{20}{3}}{\binom{30}{5}} = 0.360$$

NOTE: This result can be obtained using a conditional probability argument; consider event $F \subseteq E$, where F = "sequence of objects 11222 obtained". Then

$$F = \bigcap_{i=1}^{5} F_{ij}$$

where F_{ij} = "type j object obtained on draw i" $i = 1, ..., 5, j = 1, 2$. Then

$$P(F) = P(F_{11})P(F_{21}|F_{11})...P(F_{52}|F_{11}, F_{21}, F_{32}, F_{42}) = \frac{10}{30}\frac{9}{29}\frac{20}{28}\frac{19}{27}\frac{18}{26}$$

Now consider event G where $G =$ "sequence of objects 12122 obtained". Then

$$P(G) = \frac{10}{30}\frac{20}{29}\frac{9}{28}\frac{19}{27}\frac{18}{26}$$

i.e. P(G) =P(F). In fact, **any** sequence containing two Type I and three Type II objects has this probability, and there are $\binom{5}{2}$ such sequences. Thus, as all such sequences are mutually exclusive,

$$P(E) = \binom{5}{2}\frac{10}{30}\frac{9}{29}\frac{20}{28}\frac{19}{27}\frac{18}{26} = \frac{\binom{10}{2}\binom{20}{3}}{\binom{30}{5}}$$

as before.

EXAMPLE 3. Sampling with replacement.

Consider an Urn Model with 10 Type I objects and 20 Type II objects, and an experiment involving sampling five objects with replacement. Let $E =$ "precisely 2 Type I objects selected". Again, we need to calculate N and $n(E)$ in order to calculate P(E). In this case N is the number of ways of choosing 5 from 30 items with replacement, and hence

$$N = 30^5$$

To calculate $n(E)$, we think of E occurring by first choosing 2 Type I objects from 10, and 3 Type II objects from 20 in any order. Consider such sequences of selection

Sequence	Number of ways
11222	10.10.20.20.20
12122	10.20.10.20.20
.	.

etc., and thus a sequence with 2 Type I objects and 3 Type II objects can be obtained in $10^2 20^3$ ways. As before there are $\binom{5}{2}$such sequences, and thus

$$P(E) = \frac{\binom{5}{2}10^2 20^3}{30^5} = 0.329.$$

Again, this result can be obtained using a conditional probability argument; consider event $F \subseteq E$, where $F =$ "sequence of objects 11222 obtained". Then

$$P(F) = \left(\frac{10}{30}\right)^2\left(\frac{20}{30}\right)^3$$

as the results of the draws are **independent**. This result is true for any sequence containing two Type I and three Type II objects, and there are $\binom{5}{2}$ such sequences that are mutually exclusive, so

$$P(E) = \binom{5}{2}\left(\frac{10}{30}\right)^2\left(\frac{20}{30}\right)^3$$