EXAM-STYLE QUESTIONS

(taken from 2001 Assessed Coursework)

1. (a) Continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(x,y) = c_1(x+y) \qquad\qquad 0 \le x, y \le 1$$

and zero otherwise, for constant c_1. Find

(i) the value of c_1

(ii) the marginal pdf of X, f_X

(iii) the probability

$$\mathrm{P}\left[Y < \frac{1}{2}\right]$$

(iv) the probability

$$\mathrm{P}\left[Y < X^2\right]$$

(b) Now suppose that the range (and hence the joint distribution) of (X, Y) is changed so that

$$0 \le x, y \le 1 \textbf{ and } 0 \le x + y \le 1$$

with joint pdf

$$f_{X,Y}(x,y) = c_2(x+y)$$

for constant c_2.

Evaluate $\mathrm{P}\left[Y < X^2\right]$ for this new specification - you may leave your answer in terms of

$$\alpha = \frac{-1+\sqrt{5}}{2}$$

[Hint: sketch the regions of interest in (iii), (iv) and (b).]

2. Now suppose that the random variables X and Y have joint pdf specified as

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(x^2+y^2\right)\right\} \qquad\qquad -\infty \le x, y \le \infty$$

and suppose that X and Y correspond to the **Cartesian** x- and y-coordinates of a (random) point in the plane.

(a) Suppose that the **polar** coordinates of the point (radius, angle measured from the positive real axis) are random variables (R, T). Find the joint pdf of (R, T), $f_{R,T}$.

(b) Consider random variables

$$U = \exp\left\{-\frac{R^2}{2}\right\} \qquad V = \frac{T}{2\pi}$$

Are random variables U and V independent ? Justify your answer.

SOLUTIONS

1(a) We have

$$f_{X,Y}(x,y) = c_1(x+y) \qquad 0 \le x, y \le 1$$

and zero otherwise.

(i) To compute c_1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

and as

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= \int_0^1\int_0^1 f_{X,Y}(x,y)dxdy = \int_0^1\int_0^1 c_1(x+y)dxdy \\ &= c_1\int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy \\ &= c_1\int_0^1 \left(\frac{1}{2} + y\right) dy = c_1\left[\frac{y}{2} + \frac{y^2}{2}\right]_0^1 = c_1\end{aligned}$$

so that $c_1 = 1$.

(ii) To compute the marginal for X, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \qquad 0 \le x \le 1$$

Note that, by symmetry, we also have

$$f_Y(y) = y + \frac{1}{2} \qquad 0 \le y \le 1$$

(iii) Using the result from (ii),

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_0^y \left(t + \frac{1}{2}\right) dt = \left[\frac{t^2}{2} + \frac{t}{2}\right]_0^y = \frac{y^2}{2} + \frac{y}{2} = \frac{y}{2}(1+y) \qquad 0 \le y \le 1$$

so that

$$P\left[Y < \frac{1}{2}\right] = F_Y\left(\frac{1}{2}\right) = \frac{3}{8}$$

(iv) Can write

$$P\left[Y < X^2\right] = \int\int_A f_{X,Y}(x,y)\,dxdy$$

where A is the set

$$A \equiv \left\{(x,y) : y < x^2\right\}$$

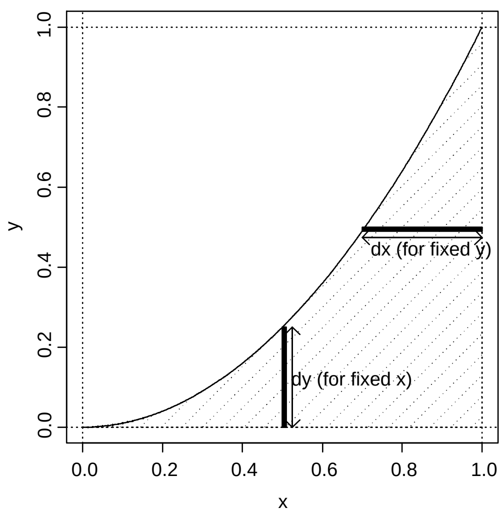
so that, integrating dx first (for fixed y, on the range from $\left(\sqrt{y}, 1\right)$)

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_0^1 \left\{ \int_{\sqrt{y}}^1 (x+y)dx \right\} dy = \int_0^1 \left\{ \left[\frac{x^2}{2} + xy \right]_{\sqrt{y}}^1 \right\} dy \\
&= \int_0^1 \left\{ \frac{1}{2} + y - \frac{y}{2} - \sqrt[3/2]{y} \right\} dy = \int_0^1 \left\{ \frac{1}{2} + \frac{y}{2} - \sqrt[3/2]{y} \right\} dy \\
&= \left[\frac{y}{2} + \frac{y^2}{4} - \frac{2}{5}\sqrt[5/2]{y} \right]_0^1 = \frac{1}{2} + \frac{1}{4} - \frac{2}{5} = \frac{7}{20}
\end{aligned}$$

or, equivalently, integrating dy first (for fixed x, on the range from $\left(0, x^2\right)$)

$$P\left[Y < X^2\right] = \int_0^1 \left\{ \int_0^{x^2} (x+y)dy \right\} dx = \int_0^1 \left\{ \left[xy + \frac{y^2}{2} \right]_0^{x^2} \right\} dx = \int_0^1 \left\{ x^3 + \frac{x^4}{2} \right\} dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^1$$

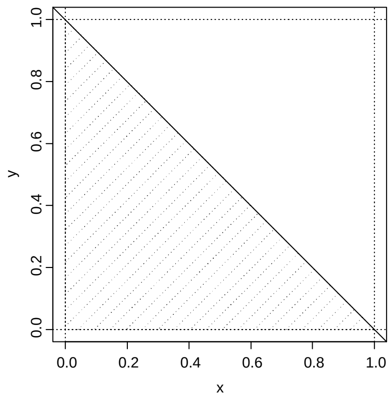
and hence $P\left[Y < X^2\right] = \dfrac{1}{4} + \dfrac{1}{10} = \dfrac{7}{20}$.



(b) Under the new specification, a change in the range of integration changes the normalization constant from c_1 to c_2 and also the value $P\left[Y < X^2\right]$. First, to compute c_2, we integrate dy for fixed x; for any fixed x the range of integration dy is from the axis to the line $x+y=1$, that is, from $y=0$ to $y=1-x$ and hence

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= \int_0^1\left\{\int_0^{1-x} f_{X,Y}(x,y)dy\right\}dx = \int_0^1\left\{\int_0^{1-x} c_2(x+y)dy\right\}dx \\
&= c_2\int_0^1\left[xy+\frac{y^2}{2}\right]_0^{1-x} dx \\
&= c_2\int_0^1\left(x(1-x)+\frac{(1-x)^2}{2}\right)dx = \frac{c_2}{2}\int_0^1\left(2x(1-x)+(1-x)^2\right)dx \\
&= \frac{c_2}{2}\int_0^1\left(1-x^2\right)dx = \frac{c_2}{2}\left[x-\tfrac{x^3}{3}\right]_0^1 = \frac{c_2}{3}
\end{aligned}$$

so that $c_2 = 3$.



Now, to calculate $P\left[Y < X^2\right]$ first note that the region A of interest is the region bounded by the horizontal axis and the lines $x+y=1$ and $y=x^2$. These lines meet when

$$x^2 = 1-x \therefore x^2+x-1=0 \Longleftrightarrow x = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

and we require the root that lies in the range of X, that is

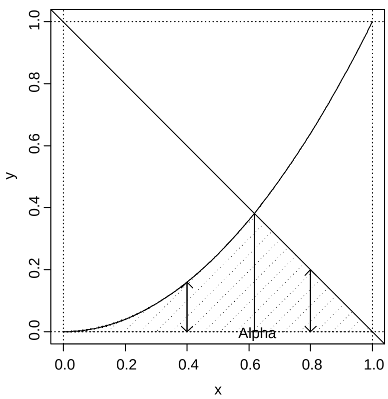
$$\frac{-1+\sqrt{5}}{2} = \alpha \qquad \text{say.}$$

Now, by inspection of a suitable sketch, we see that again integration dy for fixed x, we need to split the range of x into two ranges, that is, first, from 0 to α and second from α to 1, as the range of integration

dy is **different** in these two cases.

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_A \int f_{X,Y}(x,y)dydx = \int_0^\alpha \left\{ \int_0^{x^2} 3(x+y)dy \right\} dx + \int_\alpha^1 \left\{ \int_0^{1-x} 3(x+y)dy \right\} dx \\
&= 3\int_0^\alpha \left\{ \left[xy + \frac{y^2}{2} \right]_0^{x^2} \right\} dx + 3\int_\alpha^1 \left\{ \left[xy + \frac{y^2}{2} \right]_0^{1-x} \right\} dx \\
&= 3\int_0^\alpha \left\{ x^3 + \frac{x^4}{2} \right\} dx + 3\int_\alpha^1 \left\{ x(1-x) + \frac{(1-x)^2}{2} \right\} dx \\
&= 3\left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^\alpha + \frac{3}{2}\left[x - \frac{x^3}{3} \right]_\alpha^1 = 3\left(\frac{\alpha^4}{4} + \frac{\alpha^5}{10} \right) + \frac{3}{2}\left(\frac{2}{3} - \left(\alpha - \frac{\alpha^3}{3} \right) \right) \\
&= \tfrac{3}{20}\left(5\alpha^4 + 2\alpha^5\right) + \left(1 - \frac{3}{2}\alpha + \frac{\alpha^3}{2} \right) = \tfrac{1}{20}(6\alpha^5 + 15\alpha^4 + 10\alpha^3 - 30\alpha + 20)
\end{aligned}$$

which gives $P\left[Y < X^2\right] = 0.3274$.



Alternatively, integrating dx first for fixed y

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_A \int f_{X,Y}(x,y)dxdy = \int_0^{\alpha^2} \left\{ \int_{\sqrt{x}}^{1-x} 3(x+y)dy \right\} dx = 3\int_0^{\alpha^2} \left\{ \left[xy + \frac{y^2}{2} \right]_{\sqrt{x}}^{1-x} \right\} dx \\
&= 3\int_0^{\alpha^2} \left\{ x(1-x) + (1-x)^2 - x^{3/2} - \frac{x}{2} \right\} dx \\
&= 3\left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{(1-x)^3}{3} - \frac{2x^{5/2}}{5} - \frac{x^2}{4} \right]_0^{\alpha^2} = \tfrac{1}{20}\left(6\alpha^5 + 15\alpha^4 + 10\alpha^3 - 30\alpha + 20\right)
\end{aligned}$$

Here is a MAPLE check of the calculation : α

```
> evalf(int(int(3*(x+y),y=0..min(x^2,1-x)),x=0..1));
                              .3274575141
> a:=-1/2+sqrt(5)/2;
                         a := 1/2 sqrt(5) - 1/2
> alpha:=convert(a,float);
                           alpha := .6180339890
> (3/20)*(5*alpha^4+2*alpha^5)+(1-3*alpha/2+alpha^3/2);
                              .3274575144
```

(b) The multivariate transformation result for two variables is to be used. First, we have that

$$\left.\begin{aligned} R &= \sqrt{X^2+Y^2} \\ T &= \tan^{-1}\left(\frac{Y}{X}\right) \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} X &= R\cos T \\ Y &= R\sin T \end{aligned}\right.$$

so that the range of R is $(0,\infty)$ and the range of T is $(0,2\pi)$, and the Jacobian is

$$J(r,t) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r\sin^2 t + r\cos^2 t = r$$

Hence, by the transformation result

$$f_{R,T}(r,t) = f_{X,Y}(r\cos t, r\sin t)\, J(r,t) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(r^2\cos^2 t + r^2\sin^2 t\right)\right\} r = \frac{1}{2\pi} r\exp\left\{-\frac{r^2}{2}\right\}$$

for $0<r$, $0<t<2\pi$, and zero otherwise. Now, the marginal pdf for R, f_R is given by

$$f_R(r) = \int_{-\infty}^{\infty} f_{R,T}(r,t)dt = \int_0^{2\pi} \frac{1}{2\pi} r\exp\left\{-\frac{r^2}{2}\right\} dt = \frac{1}{2\pi} r\exp\left\{-\frac{r^2}{2}\right\}\int_0^{2\pi} dt = r\exp\left\{-\frac{r^2}{2}\right\}$$

for $0<r$ and zero otherwise. Similarly the marginal pdf for T, f_T is given by

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{R,T}(r,t)dr = \int_0^{\infty} \frac{1}{2\pi} r\exp\left\{-\frac{r^2}{2}\right\} dr \\ &= \frac{1}{2\pi}\int_0^{\infty} r\exp\left\{-\frac{r^2}{2}\right\} dr \\ &= \frac{1}{2\pi}\left[-\exp\left\{-\frac{r^2}{2}\right\}\right]_0^{\infty} = \frac{1}{2\pi} \end{aligned}$$

for $0<t<2\pi$, and zero otherwise. Hence

$$f_{R,T}(r,t) = f_R(r) f_T(t)$$

and the joint range is a Cartesian product of the range of each variable, and so R and T are independent random variables.

(ii) The joint pdf of variables U and V is obtained as follows. First, note

$$\left.\begin{aligned} U &= \exp\left\{-\tfrac{R^2}{2}\right\} \\ V &= \frac{T}{2\pi} \end{aligned}\right\} \iff \left\{\begin{aligned} R &= \sqrt{-2\log U} \\ T &= 2\pi V \end{aligned}\right.$$

so that the range of U and V is $(0,1)$ and the Jacobian is

$$J(u,v) = \begin{vmatrix} \dfrac{\partial r}{\partial u} & \dfrac{\partial r}{\partial v} \\ \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{-1}{U\sqrt{-2\log U}} & 0 \\ 0 & 2\pi \end{vmatrix} = \frac{2\pi}{U\sqrt{-2\log U}}$$

Hence, by the transformation result

$$f_{U,V}(u,v) = f_{R,T}\left(\sqrt{-2\log u}, 2\pi v\right) J(u,v) = f_R\left(\sqrt{-2\log u}\right) f_T(2\pi v) J(u,v) \qquad 0 < u, v < 1$$

which gives

$$f_{U,V}(u,v) = \sqrt{-2\log u} \exp\left\{-\frac{-2\log u}{2}\right\} \cdot \frac{1}{2\pi} \cdot \frac{2\pi}{u\sqrt{-2\log u}} = 1 \qquad 0 < u, v < 1$$

and hence

$$f_{U,V}(u,v) = f_U(u) f_V(v) \qquad 0 < u, v < 1$$

and U and V are independent (and each is $Uniform(0,1)$).

Note that this full calculation can be circumvented by noting that U is a function of R only, and V is a function of T only, and hence as R and T are independent then U and V are also automatically independent. Note also that the transformation from R to U and the transformation from T to V are both transformations via functions that are the cdfs of the original variable, that is,

$$U = F_R(R) \qquad\qquad V = F_T(T)$$

and therefore, by the result given in lectures, we must have that U and V are $Uniform(0,1)$)