DISCRETE DISTRIBUTIONS
WORKED EXAMPLES

EXAMPLE: LIGHTNING DAMAGE

For insurance purposes, the amount of damage (in £1000) that will be caused to a TV mast by lightning during the next calendar year is to be assessed. Let N be a discrete random variable taking values on $\{0, 1, 2, ...\}$ that records the number of lightning strikes. Historical information indicates that an appropriate probability model for N has pmf

$$f_N(n) = \frac{e^{-2}2^n}{n!} \qquad n = 0, 1, 2, ...$$

(that is, a *Poisson* distribution with parameter 2; we write $N \sim Poisson(2)$). Historical information, and economic predictions, also indicate that for the next calendar year, the amounts of damage caused by the strikes will themselves be discrete random variables with range $\{0, 1, 2, ...\}$ and with pmf f_X, Let X_i be the random variable corresponding to strike i. It may also be assumed that the random variables are *mutually independent*, that is, for any n,

$$P\left[\bigcap_{i=1}^{n}(X_i = x_i)\right] = P\left[X_1 = x_1, X_2 = x_2, ..., X_n = x_n\right] = \prod_{i=1}^{n} P\left[X_i = x_i\right]$$

What is the probability distribution of

(i) the total amount of damage caused, S

(ii) the minimum amount of damage caused by a strike, M

(iii) the maximum amount of damage caused by a strike, L

SOLUTION :

(i) For the next calendar year, the number of strikes N is not observed. However, if $N = n$ (for some non-negative integer n), then let discrete random variable S_n be the total amount of damage caused by the n strikes. Then, *conditional on* $N = n$,

$$S_n = X_1 + X_2 + ... + X_n = \sum_{i=1}^{n} X_i$$

that is, the sum of n independent random variables that each have pmf f_X. The random variable S_n also has range $\{0, 1, 2, ...\}$. We are actually interested in the random variable S, not conditional on any particular value of N.

Let the pmfs of S_n and S be f_{S_n} and f_S respectively. Them for $s \in \{0, 1, 2, ...\}$, by the **Theorem of Total Probability**, using a **partition** constructed using the different possible values of N, we have

$$f_S(s) = P[S = s] = \sum_{n=0}^{\infty} P[S = s|N = n]P[N = n]$$

(regarding the sample space Ω as being composed of pairs of values of (S, N)). Now

$$P[S = s|N = n] \equiv P[S_n = s|N = n] = f_{S_n}(s) \qquad \therefore \qquad f_S(s) = \sum_{n=0}^{\infty} f_{S_n}(s) f_N(n). \qquad (1)$$

We do not have $f_{S_n}(s)$ directly, but it could be computed by again using the Theorem of Total Probability, that is, for $n=2$, $S_2 = X_1 + X_2$ and, partitioning using different possible values of X_1

$$f_{S_2}(s) = \sum_{x=0}^{s} P\left[S_2 = s | X_1 = x\right] P\left[X_1 = x\right] = \sum_{x=0}^{s} P\left[X_2 = s - x\right] P\left[X_1 = x\right] = \sum_{x=0}^{s} f_{X_2}(s-x) f_{X_1}(x).$$

The distribution of S_n can be computed recursively in this way, but is quite laborious.

A preferable method of computation uses *probability generating functions* (pgfs). For a discrete random variable X taking values on range $\{0, 1, 2, ...\}$, the pgf G_X is defined for real value t by

$$G_X(t) = \sum_{x=0}^{\infty} t^x f_X(x)$$

and a key result tells us that if X_1 and X_2 are independent, and $Y = X_1 + X_2$ then the pgf of Y is given by

$$G_Y(t) = G_{X_1}(t) G_{X_2}(t).$$

Here, by extension, we have

$$S_n = \sum_{i=1}^{n} X_i \qquad \Longrightarrow G_{S_n}(t) = G_{X_1}(t) G_{X_2}(t) ... G_{X_n}(t) = \{G_X(t)\}^n \tag{2}$$

as $X_1, ..., X_n$ are identically distributed, and have the same pgf. Now, considering the pgf of S, we have

$$\begin{aligned}
G_S(t) &= \sum_{s=0}^{\infty} t^s f_S(s) = \sum_{s=0}^{\infty} t^s \left\{ \sum_{n=0}^{\infty} f_{S_n}(s) f_N(n) \right\} && \text{from (1)} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{\infty} t^s f_{S_n}(s) \right\} f_N(n) = \sum_{n=0}^{\infty} \{G_{S_n}(t)\} f_N(n) && \text{exchanging the summation order} \\
&= \sum_{n=0}^{\infty} \{G_X(t)\}^n f_N(n) && \text{from (2)} \\
&= \sum_{n=0}^{\infty} \{G_X(t)\}^n \frac{e^{-2} 2^n}{n!} \\
&= e^{-2} \sum_{n=0}^{\infty} \frac{\{2 G_X(t)\}^n}{n!} = e^{-2} \exp\{2 G_X(t)\} && \text{summing an exponential series}
\end{aligned}$$

Note that this result says precisely that, recalling the expectation definition of pgfs, $G_S(t) = E_{f_S}\left[t^S\right]$, and

$$G_S(t) = E_{f_N}\left[G_{S_n}(t)\right] = E_{f_N}\left[\{G_X(t)\}^n\right] = E_{f_N}\left[\left\{E_{f_X}\left[t^X\right]\right\}^n\right] = G_N\left(G_X(t)\right).$$

Hence

$$G_S(t) = \exp\{-2 + 2 G_X(t)\} = \exp\{2\left(G_X(t) - 1\right)\}$$

and subsequently the pmf of S (and other quantities, such as expectations and other moments can be derived easily.

(ii) For the minimum strike-related damage, we use the same partitioning method and first condition on $N = n$. Define

$$M_n = \min\{X_1, ..., X_n\}$$

as the minimum value obtained from n strikes. We have that

$$P[M_n > x|N = n] = P\left[\bigcap_{i=1}^{n}(X_i > x_i)|N = n\right] = P[X_1 > x, ..., X_n > x|N = n] = \prod_{i=1}^{n} P[X_i > x]$$

by independence. Define $r_X(x)$ as the *reliability function* for X, that is, $r_X(x) = P[X > x] = 1 - F_X(x)$, then

$$P[M_n > x|N = n] = \prod_{i=1}^{n} P[X_i > x] = \prod_{i=1}^{n} r_X(x) = \{r_X(x)\}^n.$$

Then, unconditionally, following the same calculation as in (i),

$$\begin{aligned} r_M(x) = P[M > x] &= \sum_{n=0}^{\infty} P[M_n > x|N = n]P[N = n] = \sum_{n=0}^{\infty} \{r_X(x)\}^n f_N(n) \\ &= \sum_{n=0}^{\infty} \{r_X(x)\}^n \frac{e^{-2}2^n}{n!} = e^{-2}\sum_{n=0}^{\infty} \frac{\{2r_X(x)\}^n}{n!} = e^{-2}\exp\{2r_X(x)\} \end{aligned}$$

Hence

$$r_M(x) = \exp\{2(r_X(x) - 1)\} \Leftrightarrow F_M(x) = 1 - r_M(x) = 1 - \exp\{2(r_X(x) - 1)\}$$

(iii) For the maximum strike-related damage, we use the same partitioning method and first condition on $N = n$. Define

$$L_n = \max\{X_1, ..., X_n\}$$

as the maximum value obtained from n strikes. We have that

$$P[L_n \leq x|N = n] = P\left[\bigcap_{i=1}^{n}(X_i \leq x_i)|N = n\right] = P[X_1 \leq x, ..., X_n \leq x|N = n] = \prod_{i=1}^{n} P[X_i \leq x]$$

by independence. Then, by inspecting the cdf of L

$$P[L_n \leq x|N = n] = \prod_{i=1}^{n} P[X_i \leq x] = \prod_{i=1}^{n} F_X(x) = \{F_X(x)\}^n.$$

Then, unconditionally, following the same calculation as in (i),

$$\begin{aligned} F_L(x) = P[L \leq x] &= \sum_{n=0}^{\infty} P[L_n \leq x|N = n]P[N = n] = \sum_{n=0}^{\infty} \{F_X(x)\}^n f_N(n) \\ &= \sum_{n=0}^{\infty} \{F_X(x)\}^n \frac{e^{-2}2^n}{n!} = e^{-2}\sum_{n=0}^{\infty} \frac{\{2F_X(x)\}^n}{n!} = e^{-2}\exp\{2F_X(x)\} \end{aligned}$$

Hence

$$F_L(x) = \exp\{2(F_X(x) - 1)\}$$

EXAMPLE: POPULATION EXTINCTION/GENEALOGY

("Branching Process" Example from Chapter 1)

Let X be the number of offspring of an animal in some population; X is a discrete random variable and by assumption

$$P[X=0]>0.$$

Let $Z_1, Z_2, Z_3...$ be the numbers of animals in the 1st, 2nd, 3rd etc. generations (and, for completeness, define $Z_0 = 1$). Note that

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{(n)}$$

where $X_i^{(j)}$ is the number of offspring of the ith animal in the jth population, a random variable with the same distribution as X. Finally, let

$$P\left[X_i^{(j)} = k\right] = p_k$$

where$\{p_0, p_1, p_2, ...\}$ is a probability distribution yet to be specified. Suppose that G_X is the probability generating function of X.

Under these assumptions, what is the probability of ultimate extinction ?

SOLUTION :

We consider first the probability generating function of Z_n. By definition

$$\begin{aligned}
G_{Z_n}(t) &= \sum_{z=0}^{\infty} t^z f_{Z_n}(z) = \sum_{z=0}^{\infty} t^z P[Z_n = z] \\
&= \sum_{z=0}^{\infty} t^z \left\{ \sum_{y=0}^{\infty} P[Z_n = z | Z_{n-1} = y] P[Z_{n-1} = y] \right\} && \text{by the Theorem of Total Prob.} \\
&= \sum_{y=0}^{\infty} \left\{ \sum_{z=0}^{\infty} t^z P[Z_n = z | Z_{n-1} = y] \right\} P[Z_{n-1} = y] && \text{exchanging the summation order} \\
&= \sum_{n=0}^{\infty} \left\{ G_{Z_n|Z_{n-1}}(t|y) \right\} f_{Z_{n-1}}(y)
\end{aligned}$$

where $G_{Z_n|Z_{n-1}}(t|y)$ is the conditional generating function of Z_n, given $Z_{n-1} = y$, and $f_{Z_{n-1}}(.)$ is the pmf for Z_{n-1}. Now, given $Z_{n-1} = y$, the sum

$$\sum_{i=1}^{Z_{n-1}} X_i^{(n-1)} = \sum_{i=1}^{y} X_i^{(n-1)}$$

is a sum of y independent variables with the same distribution. Hence, by the result from the previous example

$$G_{Z_n|Z_{n-1}}(t|y) = \{G_X(t)\}^y$$

and thus

$$G_{Z_n}(t) = \sum_{y=0}^{\infty} \{G_X(t)\}^y f_{Z_{n-1}}(y) = G_{Z_{n-1}}(G_X(t)) \tag{3}$$

which gives a recursive method for computing G_{Z_n}. Also, by recursion

$$G_{Z_n}(t) = G_{Z_{n-1}}(G_X(t)) = G_{Z_{n-2}}(G_X(G_X(t))) = ... = G_X(G_X(G_X(...G_X(G_X(t)))))$$

that is, ultimately, an n-fold computation. Taking the final expression, we have, by considering the internal $n-1$ terms,

$$G_{Z_n}(t) = G_X(G_X(G_X(...G_X(G_X(t))))) = G_X\left(G_{Z_{n-1}}(t)\right) \tag{4}$$

Denote by π_n the probability $P[Z_n = 0]$. Then, by definition of the pgf, $\pi_n = G_{Z_n}(0)$, that is the coefficient of t^0 in the expansion of G_{Z_n}.and hence by (4)

$$\pi_n = G_{Z_n}(0) = G_X\left(G_{Z_{n-1}}(0)\right) = G_X(\pi_{n-1}) \tag{5}$$

Now define the probability of extinction π

$$\pi = P[Z_m = 0 \text{ for some } m] = \lim_{n\to\infty} \pi_n$$

It follows that π (if it exists) is the solution of (5), that is

$$\pi = G_X(\pi) \tag{6}$$

Now, the function $G_X(t)$ is continuous and differentiable on the range $(0,1)$. Also, as it is a convex combination, via f_X, of terms $1, t, t^2, t^3, ...$ and hence is non-decreasing on $(0,1)$; for $t_1 \leq t_2$

$$G_X(t_1) = \sum_{x=0}^{\infty} t_1^x f_X(x) \leq \sum_{x=0}^{\infty} t_2^x f_X(x) = G_X(t_2)$$

Note that also for $t_1 \leq t_2$

$$G'_X(t_1) = \frac{d}{dt} G_X(t)|_{t=t_1} = \sum_{x=0}^{\infty} x t_1^{x-1} f_X(x) \leq \sum_{x=0}^{\infty} x t_2^{x-1} f_X(x) = G'_X(t_2)$$

so $G_X(t)$ also has a slope that is non-decreasing in t. Finally,

$$G_X(0) = P[X = 0] = p_0 > 0 \qquad\qquad G_X(1) = \sum_{x=0}^{\infty} 1^x f_X(x) = \sum_{x=0}^{\infty} f_X(x) = 1$$

and the slope of $G_X(t)$ at $t = 1$ is

$$G'_X(1) = \frac{d}{dt} G_X(t)|_{t=1} = \sum_{x=0}^{\infty} x 1^{x-1} f_X(x) = \sum_{x=0}^{\infty} x f_X(x) = E_{f_X}[X] = \mu, \text{ say.}$$

Now reconsider (6); to find π we seek the solution of the equation

$$x = G_X(x) \qquad \text{or} \qquad x - G_X(x) = 0.$$

It is clear from the diagrams below that if $G'_X(1) > 1$, the equation has a unique solution away from 1, but if $G'_X(1) \leq 1$, the only solution is $\pi = 1$. Therefore, as

$$G'_X(1) = E_{f_X}[X] = \mu$$

we can observe that the population becomes extinct with probability $\pi < 1$ if $E_{f_X}[X] = \mu > 1$, but becomes extinct with probability $\pi = 1$ if $E_{f_X}[X] = \mu \leq 1$.

For a concrete example, let

$$f_X(x) = \theta^x (1-\theta) \qquad x = 0, 1, 2, 3, ...$$

for some θ $(0 < \theta < 1)$. Then $p_0 = P[X = 0] = 1 - \theta$ and

$$\begin{aligned}
\sum_{x=0}^{\infty} f_X(x) &= \sum_{x=0}^{\infty} \theta^x (1-\theta) = (1-\theta) \sum_{x=0}^{\infty} \theta^x = \frac{1-\theta}{1-\theta} = 1 \\
G_X(t) &= \sum_{x=0}^{\infty} t^x f_X(x) = \sum_{x=0}^{\infty} t^x \theta^x (1-\theta) = (1-\theta) \sum_{x=0}^{\infty} t^x \theta^x = \frac{1-\theta}{1-\theta t} \qquad (0 < t < 1) \\
G'_X(t) &= \frac{\theta(1-\theta)}{(1-\theta t)^2} \Longrightarrow G'_X(1) = \frac{\theta(1-\theta)}{(1-\theta)^2} = \frac{\theta}{1-\theta}
\end{aligned}$$

so that $G'_X(1) = E_{f_X}[X] = \dfrac{\theta}{1-\theta} = \mu$. Finally, for π, from (6),

$$\pi - G_X(\pi) = 0 \Longrightarrow \pi - \frac{1-\theta}{1-\theta\pi} = 0 \Longrightarrow \pi = 1 \text{ or } \pi = \frac{1-\theta}{\theta}$$

Figure 1: Extinction probability calculation if $P[X = x] = \theta^x (1-\theta)$

EXAMPLE

Suppose that two systems are set in operation on day 1. The probability that system 1 fails for the first time on day n is denoted $p_1(n)$ (that is, the conditional probability, defined as a function of n, that the system fails on day n *given* that it has not failed on any preceding day); similarly, the probability that system 2 fails for the first time on day n is denoted $p_2(n)$ Let X_1 and X_2 be the discrete random variables corresponding to the days on which system 1 and system 2 fail for the first time, respectively.

Show that the probability mass functions of X_1 and X_2 are given by

$$f_{X_i}(n) = p_i(n) \prod_{k=1}^{n-1} (1 - p_i(k)) \qquad n = 1, 2, ... \quad \text{for } i = 1, 2$$

and zero elsewhere, respectively.

Suppose that, $p_1(.)$ and $p_2(.)$ are specified by

$$\begin{aligned} p_1(n) &= \frac{1}{n+1} && n = 1, 2, ... \\ p_2(n) &= 1 - e^{-\lambda n} && n = 1, 2, ... \end{aligned}$$

where λ is a positive real constant, so that p_1 is decreasing and p_2 is increasing with n.

Show that p_1 and p_2 lead to valid probability models for X_1 and X_2, and find the mass and distribution functions of X_1 and X_2.

SOLUTION :

Let D_k denote the event that system 1 fails on day k, for $k = 1, 2, ...$ Then, if S_n denotes the event that the first failure occurs on day n then

$$S_n = D_1' \cap D_2' \cap D_3' \cap ... \cap D_{n-1}' \cap D_n$$

so that, by the chain rule

$$\begin{aligned} P(S_n) &= P(D_1')\, P(D_2'|D_1') P(D_3'|D_1' \cap D_2') ... P(D_{n-1}'|D_1' \cap D_2' \cap D_3' \cap ... \\ &\quad ... \cap D_{n-2}') P(D_n|D_1' \cap D_2' \cap D_3' \cap ... \cap D_{n-1}') \\ &= (1 - p_1(1))(1 - p_1(2))(1 - p_1(3)) ... (1 - p_1(n-1)) p_1(n) \end{aligned}$$

as for each k, we have

$$P(D_k|D_1' \cap D_2' \cap D_3' \cap ... \cap D_{k-1}') = p_1(k) \qquad P(D_k'|D_1' \cap D_2' \cap D_3' \cap ... \cap D_{k-1}') = 1 - p_1(k)$$

as these conditional probabilities are given in the question. Hence, for $i = 1, 2$,

$$f_{X_i}(n) = P[X_i = n] = P(S_n) = p_i(n) \prod_{k=1}^{n-1} (1 - p_i(k)) \qquad n = 1, 2, ...$$

Now, we also have, using the chain rule, that

$$P[X_i > n] = P(D_1' \cap D_2' \cap D_3' \cap ... \cap D_{n-1}' \cap D_n') = \prod_{k=1}^{n} (1 - p_i(k))$$

so for system 1 we have

$$P[X_1 > n] = \prod_{k=1}^{n}(1 - p_1(k)) = \prod_{k=1}^{n}\left(1 - \frac{1}{k+1}\right) = \prod_{k=1}^{n}\left(\frac{k}{k+1}\right) = \frac{1}{n+1}$$

and hence the cdf of X_1 is given by

$$F_{X_1}(n) = P[X_1 \leq n] = 1 - P[X_1 > n] = \frac{n}{n+1} \qquad n = 1, 2, ...$$

which is a valid cdf, as it takes values on $[0, 1]$, is non-decreasing in n, and is a right-continuous (step) function. By differencing, we have

$$\begin{aligned} f_{X_1}(n) &= P[X_1 \leq n] - P[X_1 \leq n-1] \\ &= (1 - P[X_1 > n]) - (1 - P[X_1 > n-1]) \\ &= \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right) \\ &= \frac{1}{n(n+1)} \qquad\qquad n = 1, 2, 3, ... \end{aligned}$$

Similarly, for system 2,

$$P[X_2 > n] = \prod_{k=1}^{n}(1 - p_2(k)) = \prod_{k=1}^{n} e^{-\lambda k} = \exp\left\{-\lambda\sum_{k=1}^{n} k\right\} = \exp\left\{-\frac{\lambda}{2}n(n+1)\right\}$$

and thus

$$F_{X_2}(n) = P[X_2 \leq n] = 1 - P[X_2 > n] = 1 - \exp\left\{-\frac{\lambda}{2}n(n+1)\right\} \qquad n = 1, 2, ..$$

which is again a valid cdf. Finally, we have

$$\begin{aligned} f_{X_2}(n) &= P[X_2 \leq n] - P[X_2 \leq n-1] \\ &= (1 - P[X_2 > n]) - (1 - P[X_2 > n-1]) \\ &= \left(1 - \exp\left\{-\frac{\lambda}{2}n(n+1)\right\}\right) - \left(1 - \exp\left\{-\frac{\lambda}{2}n(n-1)\right\}\right) \\ &= \left(1 - e^{-\lambda n}\right)\exp\left\{-\frac{\lambda}{2}n(n-1)\right\} \qquad\qquad n = 1, 2, 3, ... \end{aligned}$$

EXAMPLE

In a cricket ball throwing contest, a competitor is permitted as many throws as they like, and the longest of their throws is recorded. The number of throws that they take is a discrete random variable, T, taking values on range $\mathbb{T} = \{1, 2, ...\}$ with probability mass function given by

$$f_T(t) = \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^{t-1} \qquad t = 1, 2, ..$$

Let random variables $X_{1,}X_2, ..., X_T$ (where, note, T is not known before the contest starts) denote the measured throws of a given competitor. Suppose that the throws are mutually independent, and have identical probability distributions specified via a "*reliability*" function r_X that specifies the probability that a single throw exceeds a given value, that is, for $x > 0$,

$$r_X(x) = P[X_i > x] = \frac{1}{1+x} \qquad x > 0$$

for each i.

Verify that r_X specifies a valid probability model. If the number of throws taken is **known** to be t, let the longest throw recorded for a given competitor be a random variable L, so that

$$L = \max\{X_{1,}X_2, ..., X_t\},$$

find the reliability function for L, r_L say, given by

$$r_L(x) = P[L > x]$$

for $x > 0$.

Now, suppose that a competitor records a longest throw that is longer than a distance l. Find an expression for the conditional probability that the number of throws taken in total is equal to t, for $t = 1, 2, ...$

SOLUTION :

First, r_X specifies a valid probability model, as, because of the probability axioms, we require that

$$0 \le r_X(x) \le 1 \quad \text{for all } x > 0$$

and also that, here,

$$\lim_{x\to 0} r_L(x) = P[X > 0] = 1 \qquad \text{and} \qquad \lim_{x\to\infty} r_L(x) = \lim_{x\to\infty} P[X > x] = 0.$$

Now, given that $T = t$, that is that $t \ge 1$ throws were taken, we consider the event $[L \le l]$ (the event that the largest of the t throws is less than l). We have that

$$\begin{aligned}
P[L \le l\,|\,T = t] &= P[(X_1 \le l) \cap (X_2 \le l) \cap ... \cap (X_t \le l)\,|\,T = t] && \text{(no throw longer than } l\text{)} \\
&= P[X_1 \le l]P[X_2 \le l]...P[X_t \le l] && \text{(by mutual independence)} \\
&= (1 - r_X(l))(1 - r_X(l))...(1 - r_X(l)) \\
&= \left(\frac{l}{1+l}\right)^t && \text{(from above)}
\end{aligned}$$

Hence, using the theorem of total probability, using the partition given by $[T=1]\,,[T=2]\,,$...we have

$$\begin{aligned}
P[L \le l] &= \sum_{t=1}^{\infty} P[L \le l \,|\, T=t] P[T=t] \\
&= \sum_{t=1}^{\infty} \left(\frac{l}{1+l}\right)^t \left(\frac{9}{10}\right) \left(\frac{1}{10}\right)^{t-1} \\
&= \left(\frac{9}{10}\right) \left(\frac{l}{1+l}\right) \sum_{s=0}^{\infty} \left(\frac{l}{10(1+l)}\right)^s \quad \text{where } s=t-1 \\
&= \left(\frac{9}{10}\right) \left(\frac{l}{1+l}\right) \frac{1}{\left(1-\dfrac{l}{10(1+l)}\right)} \quad \text{(summing a geometric progression)} \\
&= \frac{9l}{10+9l}
\end{aligned}$$

Hence

$$r_L(l) = 1 - P[L \le l] = \frac{10}{10+9l}.$$

Now, given $[L > l]$, we have by Bayes Theorem

$$P[T=t \,|\, L>l] = \frac{P[L>l \,|\, T=t]P[T=t]}{P[L>l]} = \frac{\left\{1-\left(\dfrac{l}{l+1}\right)^t\right\}\left(\dfrac{9}{10}\right)\left(\dfrac{1}{10}\right)^{t-1}}{\left(\dfrac{10}{10+9l}\right)}$$