M2S1 - EXERCISES 5

Miscellaneous distributional results

1. The joint pdf $f_{X,Y}$ of positive random variables X and Y is specified as $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$, where $X|Y = y \sim Exponential(y)$ and $Y \sim Gamma(\alpha, \beta)$. Identify the marginal distribution of X.

2. *The Bivariate Normal Distribution*: Suppose that X_1 and X_2 are i.i.d $Normal(0,1)$ random variables. Let random variables Y_1 and Y_2 be defined by

$$\begin{aligned} Y_1 &= \mu_1 + \sigma_1\sqrt{1-\rho^2}X_1 + \sigma_1\rho X_2 \\ Y_2 &= \mu_2 + \sigma_2 X_2 \end{aligned} \qquad \text{or equivalently} \qquad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

for positive constants σ_1 and σ_2, and $|\rho| < 1$. Find the joint pdf of (Y_1, Y_2).

Show that, marginally for $i = 1, 2$, $Y_i \sim Normal\left(\mu_i, \sigma_i^2\right)$, and that conditionally

$$\begin{aligned} Y_1|Y_2 = y_2 &\sim Normal\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(y_2 - \mu_2), \sigma_1^2\left(1-\rho^2\right)\right) \\ Y_2|Y_1 = y_1 &\sim Normal\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1), \sigma_2^2\left(1-\rho^2\right)\right) \end{aligned}.$$

Find the correlation of $Y_1$and Y_2.

The Central Limit Theorem

3. Using the Central Limit Theorem, construct a Normal approximation to probability distribution of a random variable X having a

(i) Binomial distribution, $X \sim Binomial(n, \theta)$

(ii) Poisson distribution, $X \sim Poisson(\lambda)$

(iii) Negative Binomial distribution, $X \sim NegBinomial(n, \theta)$

(iv) Gamma distribution,$X \sim Gamma(\alpha, \beta)$

Limiting distributions

In the following questions, use the following results from earlier in the course; if $X_1, ...X_n$ are a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with mass function/pdf f_X and cdf F_X, let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from $X_1, ...X_n$, that is

$$Y_n = \max\{X_1, ..., X_n\} \qquad Z_n = \min\{X_1, ..., X_n\}.$$

Then the cdfs of Y_n and Z_n are given by

$$F_{Y_n}(y) = \{F_X(y)\}^n \qquad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$

4. Suppose $X_1, ..., X_n \sim Uniform(0,1)$, that is

$$F_X(x) = x \qquad 0 \leq x \leq 1$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

5. Suppose $X_1, ..., X_n \sim Exp(\lambda)$, that is

$$F_X(x) = 1 - e^{-\lambda x} \qquad x > 0$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

6. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{x} \qquad x \geq 1$$

Find the cdfs of Z_n and $U_n = Z_n^n$, and the limiting distributions of Z_n and U_n as $n \longrightarrow \infty$.

7. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}} \qquad x \in \mathbb{R}$$

Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \longrightarrow \infty$.

8. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x} \qquad x > 0$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$. Find also the cdfs of

$$U_n = Y_n/n \qquad V_n = nZ_n$$

and the limiting distributions of U_n and V_n as $n \longrightarrow \infty$.

9. Suppose $X_1, ..., X_n \sim Exp(\lambda)$. Find the cdf of $U_n = \lambda Y_n - \log n$, and the limiting distribution of U_n as $n \longrightarrow \infty$.

10. Suppose $X_1, ..., X_n \sim Bernoulli(\theta)$. Write down the mgf of X_i, the mgfs of

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad M_n = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{S_n}{n}$$

.Find the limiting form of the mgf of M_n, and hence identify the limiting distribution of M_n as $n \longrightarrow \infty$.

11. Suppose $X_1, ..., X_n \sim Poisson(\lambda)$. Let $M_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Show that

$$M_n \xrightarrow{p} \lambda$$

as $n \longrightarrow \infty$. If random variable T_n is defined by $T_n = e^{-M_n}$, show that

$$T_n \xrightarrow{p} e^{-\lambda}$$

Using the central limit theorem, find the approximate probability distribution of T_n as $n \longrightarrow \infty$.