M2S1 - EXERCISES 1

This sheet is intended to encourage revision of first year probability material, in particular the use of partitions, the **Theorem of Total Probability** *and* **Bayes Theorem**.

1. A biased coin is tossed repeatedly, with tosses mutually independent; the probability of the coin showing Heads on any toss is p. Let H_n be the event that an even number of Heads have been obtained after n tosses, let $p_n = \text{P}(\ H_n\ )$, and define $p_0 = 1$.

By conditioning on H_{n-1} and using the **Theorem of Total Probability**, show that, for $n \geq 1$,

$$p_n = (1-2p)p_{n-1} + p.$$

Find a solution to this difference equation, valid for all $n \geq 0$, of the form $p_n = A + B\lambda^n$, where A, B and λ are constants to be identified. Prove that if $p < 1/2$ then $p_n > 1/2$ for all $n \geq 1$, and find the limiting value of p_n as $n \longrightarrow \infty$. Is this limit intuitively reasonable ?

2. Suppose that N fair, standard dice are rolled, where the scores on the N dice are independent, where N is unknown. Let N_i be the event that $N = i$, for $i \geq 1$, and suppose that

$$\text{P}(N_i) = \frac{1}{2^i} \quad i = 1, 2, \ldots .$$

Let S denote the sum of the scores on the N dice, and let S_j be the event that the total score is j.

Write down an expression for the event E, corresponding to the event that the total number of dice used is even, in terms of the events N_i, $i \geq 1$.

Using the conditional probability definition, the **Theorem of Total Probability**, and **Bayes Theorem**, find the probability that

(i) $S = 4$, given that N is even. (ii) $N = 2$, given that $S = 4$.

3. *(Tricky ...)* Two players contest a match consisting of a series of games, in which the match is won by the player who first wins three games, unless the score becomes tied at two games each in which case the match continues until a player gains a lead of two games over the opponent.

Suppose that the probability that player 1 wins a game is p, that games are probabilistically independent, and that no draws are possible. Let W be the event that player 1 wins the match, and let W_i be the event that player 1 wins the match on game i, for $i \geq 1$.

Using the events W_i as a suitable *partition,* show that

$$\text{P}(W) \;=\; \frac{p^3\,(4-5p+2p^2)}{(1-2p+2p^2)} \;=\; g(p), \text{ say.}$$

[Hint: consider separately the probabilities of W_i, for the three cases $i \leq 4$, $i \geq 5$ for i even, and $i \geq 5$ for i odd. Note that, for W_i, $i \geq 5$, to occur, the game score must be two games all after four games, and that player 1 must win the last two of the i games.]

Evaluate $g(p) + g(1-p)$ and comment on this result.

4. A simple model for weather forecasting involves classifying days as either Fine or Wet, and then assuming that the weather on a given day will be the same as the weather on the preceding day with probability p. Suppose that the probability of fine weather on day indexed 1 (say Jan 1st) is denoted θ. Let θ_n denote the probability that day indexed n is Fine.

For $n = 2, 3, ...$, prove that

$$\theta_n = p\theta_{n-1} + (1-p)(1-\theta_{n-1}).$$

Using this difference equation, find θ_n explicitly as a function of n, p and θ, and find the limiting value of θ_n as $n \longrightarrow \infty$.

5. Consider two coins, of which one is normal and the other has a Head on both sides. A coin is selected and tossed n times with tosses mutually independent. Evaluate the conditional probability that the selected coin is normal, given that the first n tosses are Heads.

Now consider two coins, of which one is normal and the other is biased so that the probability of obtaining a Head is $p > 1/2$. Again, one of the coins is selected and tossed n times. Let E be the event that the n tosses result in k Heads and $n-k$ Tails, and let F be the event that the coin is fair. Find expressions for $\mathrm{P}(E)$ and $\mathrm{P}(F|E)$.

[Recall the Binomial distribution from M1S; in a sequence of n mutually independent binary (success/failure) trials, the probability of obtaining a total of k successes is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

where p is the success probability for each individual trial.]

6. The probability that a tree has n flowers is given by $(1-p)p^n$ for $n = 0, 1, 2, ...$ Each flower has probability $2/3$ of being pollinated and producing fruit, and each fruit has probability of $1/4$ of not ripening fully. It can be assumed that each developmental stage is independent of the others.

Deduce that the probability of a flower producing a ripe fruit is $1/2$. Given that a tree bears r ripe fruit, calculate the conditional probability that it originally had n flowers.
[The Negative Binomial expansion

$$(1-x)^{-(N+1)} = \sum_{i=0}^{\infty} \binom{i+N}{N} x^i \qquad |x| < 1, N > 0$$

may be useful.]

7. A company is to introduce mandatory drug testing for its employees. The test used is very accurate, in that it it gives a correct positive test (detects drugs when they are present in a blood sample) with probability 0.99, and a correct negative test (does not detect drugs when they are not present) with probability 0.98. If an individual tests positive on the first test, a second blood sample is tested. It is assumed that only 1 in 5000 employees actually does provide a blood sample with drugs present.

What is the probability that the presence of drugs in a blood sample is **detected correctly**, given

(i) a positive result on the first test (before the second test is carried out)
(ii) a positive result on both first and second tests