M2S1 : EXERCISE SHEET 3 : SOLUTIONS

1 (a) To calculate the mgf

$$\begin{aligned} M_Z(t) &= \mathrm{E}_{f_Z}[\, e^{tZ} \,] = \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \, dz = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(z-t)^2}{2}\right\} \, dz \\ &= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} \, du = e^{t^2/2} \end{aligned}$$

completing the square in z, and then setting $u = z - t$, as the integrand is a pdf.
Now, using the transformation theorem for univariate, 1-1 transformations we have $X = \mu + \dfrac{1}{\lambda} Z \iff Z = \lambda(X - \mu)$, so

$$f_X(x) = f_Z(\lambda(x-\mu)) \; \lambda = \frac{\lambda}{\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\} \qquad x \in \mathbb{R}$$

To calculate the mgf of X, use the expectation result given in lectures

$$M_X(t) = \mathrm{E}_{f_Z}\left[e^{t(\mu + Z/\lambda)}\right] = e^{\mu t} M_Z(t/\lambda) = \exp\left\{\mu t + \frac{t^2}{2\lambda^2}\right\}$$

The expectation of X is

$$\begin{aligned} \mathrm{E}_{f_X}[X] &= \int_{-\infty}^{\infty} x f_X(x) \, dx \;=\; \int_{-\infty}^{\infty} x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\} \, dx \\ &= \int_{-\infty}^{\infty} \left(\mu + t\lambda^{-1}\right) \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \lambda^{-1} \, dt \qquad t = \lambda(x-\mu) \\ &= \mu \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \, dt + \lambda^{-1} \int_{-\infty}^{\infty} t \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \, dt \\ &= \mu \end{aligned}$$

as the first integral is 1, and the second integral is zero, as the integrand is an ODD function about zero. Hence

$$\mathrm{E}_{f_X}[X] = \mu$$

and note that it is generally true that if a pdf is symmetric about a particular value, then that value is the expectation (if the expectation integral is finite). Alternately, could use the mgf result that says

$$\mathrm{E}_{f_X}[X] = \frac{d}{ds}\{M_X(s)\}_{s=0} = M_X^{(1)}(0)$$

say, so that

$$\mathrm{E}_{f_X}[X] = \frac{d}{ds}\left\{\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \left\{\left(\mu + \frac{s}{\lambda^2}\right)\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \mu$$

The expectation of $g(X) = e^X$ is

$$\begin{aligned} \mathrm{E}_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x) \, dx = \int_{-\infty}^{\infty} e^x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\} \, dx \\ &= \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1}\right\} \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \lambda^{-1} \, dt \qquad \text{setting } t = \lambda(x-\mu) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1} - \frac{t^2}{2}\right\} \, dt = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t^2 - 2t\lambda^{-1} - 2\mu\right)\right\} \, dt \end{aligned}$$

Completing the square in the exponent, we have

$$\left(t^2 - 2t\lambda^{-1} - 2\mu\right) = \left(t - \lambda^{-1}\right)^2 - \left(2\mu + \lambda^{-2}\right)$$

and hence

$$\begin{aligned}
\mathrm{E}_{f_X}\left[g(X)\right] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t-\lambda^{-1}\right)^2 + \left(\mu + \frac{1}{2\lambda^2}\right)\right\} dt \\
&= \exp\left\{\mu + \frac{1}{2\lambda^2}\right\} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(t-\lambda^{-1}\right)^2\right\} dt = \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}
\end{aligned}$$

as the integral is equal to 1, as it is the integral of a pdf for all choices of λ.

(b) If $Y = e^X$, so $\mathbb{Y} = R^+$, and from first principles we have

$$F_Y(y) = \mathrm{P}\left[Y \le y\right] = \mathrm{P}\left[e^X \le y\right] = \mathrm{P}\left[X \le \log y\right] = F_X(\log y) \qquad \Longrightarrow \qquad f_Y(y) = f_X(\log y)\,\frac{1}{y} \qquad y > 0$$

Note that the function $g(t) = e^t$ is a monotone increasing function, with $g^{-1}(t) = \log t$, so that we can use the general result directly, that is

$$f_Y(y) = f_X(g^{-1}(y))\;\; J(y) \qquad \text{where} \qquad J(y) = \left|\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y}\right| = \left|\frac{d}{dt}\left\{\log t\right\}_{t=y}\right| = \frac{1}{y}$$

Hence

$$f_Y(y) = \frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(\log y - \mu)^2\right\} \qquad y > 0.$$

For the expectation, we have from first principles

$$\begin{aligned}
\mathrm{E}_{f_Y}\left[Y\right] &= \int_0^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} y\,\frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(\log y - \mu)^2\right\} dy \\
&= \int_{-\infty}^{\infty} \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(t-\mu)^2\right\} e^t\, dt \quad = \quad \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}
\end{aligned}$$

where $t = \log y$, as the integral is precisely the one carried out above. This illustrates the transformation/expectation result that, if $Y = g(X)$, then

$$\mathrm{E}_{f_Y}\left[Y\right] = \mathrm{E}_{f_X}\left[g(X)\right]$$

(c) If $T = Z^2$, then from first principles

$$\begin{aligned}
F_T(t) &= \mathrm{P}[\ T \le t\ ] = \mathrm{P}[\ Z^2 \le t\ ] = \mathrm{P}[\ -\sqrt{t} \le Z \le \sqrt{t}\ ] \\
\Longrightarrow f_T(t) &= \frac{1}{2\sqrt{t}}\left[f_Z(\sqrt{t}) + f_Z(-\sqrt{t})\right] = \frac{1}{\sqrt{2\pi}} t^{-1/2} \exp\left\{-\frac{t}{2}\right\} \qquad t > 0
\end{aligned}$$

and hence

$$\begin{aligned}
M_T(t) &= \mathrm{E}_{f_T}[\ e^{tT}\ ] = \int_{-\infty}^{\infty} e^{tx} f_T(x)\, dx = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{x}{2}\right\} dx = \int_0^{\infty} \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{(1-2t)x}{2}\right\} dx \\
&= \left(\frac{1}{1-2t}\right)^{1/2} \int_0^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\} dy = \left(\frac{1}{1-2t}\right)^{1/2}
\end{aligned}$$

where $y = (1-2t)x$, as the integrand is a pdf.

2. By definition of mgfs for discrete variables, we can deduce immediately that, as

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x)$$

P[$X = x$] is just the coefficient of e^{tx} in the expression for M_X, and hence P[$X = 1$] $= 1/8$, P[$X = 2$] $= 1/4$ and P[$X = 3$] $= 5/8$. Also, we have $\mathrm{E}_{f_X}[\ X^r\ ] = M_X^{(r)}(0)$, so that

$$\begin{aligned} \mathrm{E}_{f_X}[\ X\ ] &= M_X^{(1)}(0) = \frac{1}{8} + 2\frac{1}{4} + 3\frac{5}{8} = \frac{5}{2} \qquad \mathrm{E}_{f_X}[\ X^2\ ] = M_X^{(2)}(0) = \frac{1}{8} + 4\frac{1}{4} + 9\frac{5}{8} = \frac{27}{4} \\ \Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] &= \mathrm{E}_{f_X}[\ X^2\ ] - \{\mathrm{E}_{f_X}[\ X\ ]\}^2 = \frac{1}{2} \end{aligned}$$

3. Can identify that $X \sim Bin(n, \theta)$, but in any case,

$$M_X(t) = (1 - \theta + \theta e^t)^n = (1 + (e^t - 1)\theta)^n = \left(1 + \theta\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right)\right)^n$$

and from the mgf definition $\mathrm{E}_{f_X}[\ X^r\ ]$ is $r!$ times the coefficient of t^r in this expansion. Difficult to identify this general term, but can easily identify the coefficient of t as $n\theta = \mathrm{E}_{f_X}[\ X\ ]$, and the coefficient of t^2 as $n\theta + n(n-1)\theta^2 = \mathrm{E}_{f_X}[\ X^2\ ]$ etc.

4. For this pdf,

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\ dx = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\ dx = e^{-2} \int_{-2}^{\infty} e^{-(1-t)x}\ dx \\ &= \frac{e^{-2}}{1-t} \int_{-2(1-t)}^{\infty} e^{-y}\ dy \ = \ \frac{e^{-2}}{1-t} \left[-e^{-y}\right]_{-2(1-t)}^{\infty} \ = \ \frac{e^{-2t}}{1-t} \qquad t < 1 \end{aligned}$$

Now

$$M_X^{(1)}(t) = \frac{e^{-2t}}{(1-t)^2}(2t-1) \qquad M_X^{(2)}(t) = \frac{e^{-2t}}{(1-t)^3}\left[1 + (2t-1)^2\right]$$

so that $M_X^{(1)}(0) = -1 = \mathrm{E}_{f_X}[\ X\ ]$ and $M_X^{(2)}(0) = 2 = \mathrm{E}_{f_X}[\ X^2\ ] \Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] = 1$

5. We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\{K_X(t)\}_{s=t} = \frac{d}{ds}\{\log M_X(t)\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathrm{E}_{f_X}[\ X\ ]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\{M_X(t)\}^2} \Longrightarrow K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\{M_X(0)\}^2} = \mathrm{E}_{f_X}[\ X^2\ ] - \{\mathrm{E}_{f_X}[\ X\ ]\}^2 =$$

and hence $K_X^{(2)}(0) = \mathrm{Var}_{f_X}[\ X\ ]$

6. Easy to see that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, with $\mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$, so X and Y are independent, where

$$f_X(x) = \sqrt{c}\,\frac{2^x}{x!} \qquad f_Y(y) = \sqrt{c}\,\frac{2^y}{y!} \qquad \text{and} \qquad \sum_{x=0}^{\infty} f_X(x) = 1 \Longrightarrow \sqrt{c} = e^{-2}$$

(marginal mass functions must have identical forms as joint mass function is symmetric in x and y) as the summation is identical to the power series expansion of e^z at $z = 2$ if $\sqrt{c} = e^{-2}$.

7. $F_{X,Y}$ is continuous and non decreasing in x and y, and

$$\lim_{x \longrightarrow -\infty} F_{X,Y}(x,y) = \lim_{y \longrightarrow -\infty} F_{X,Y}(x,y) = 0 \qquad \lim_{x,y \longrightarrow \infty} F_{X,Y}(x,y) = 1$$

so $F_{X,Y}$ is a valid cdf, and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2} \{F_{X,Y}(t_1,t_2)\}_{t_1=x,t_2=y} = \frac{e^{-x}}{\pi(1+y^2)} = f_X(x) f_Y(y)$$

so as $\mathbb{X}^{(2)} = \mathbb{R}^+ \times \mathbb{R}$, X and Y are independent.

8. The form of the joint range $\mathbb{X}^{(2)}$ is the key point; we have $\mathbb{X}^{(2)} = \{\ (x,y)\ :\ x > 0, 0 < y < \exp\{-\beta x^\alpha\}\}$, and hence

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{e^{-\beta x^\alpha}} c x^{\alpha-1}\ dy = c x^{\alpha-1} \exp\{-\beta x^\alpha\} \qquad x > 0$$

$$\Longrightarrow F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x c t^{\alpha-1} \exp\{-\beta t^\alpha\} = \frac{c}{\alpha\beta}\left(1 - \exp\{-\beta x^\alpha\}\right)$$

so that $c = \alpha\beta$. Similarly, letting $g(y) = \{-\log y/\beta\}^{1/\alpha}$, we have $0 < x < g(y)$ as $0 < y < \exp\{-\beta x^\alpha\}$, and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^{g(y)} c x^{\alpha-1}\ dx = \frac{c}{\alpha}\{g(y)\}^\alpha = -\log y \qquad 0 < y < 1$$

9. (i) If $\mathbb{X}^{(2)} = (0,1) \times (0,1)$ is the (joint) range of vector random variable (X,Y). We have

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

so that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \qquad \text{and} \qquad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the ranges of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1-y) \tag{1}$$

for some constants satisfying $c_1 c_2 = c$. Hence, the two conditions for independence are satisfied in (2), and X and Y are independent.

(ii) We must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1 \therefore c^{-1} = \int_0^1\int_0^1 x(1-y)\ dxdy = 1$$

and as

$$\int_0^1\int_0^1 x(1-y)\ dxdy = \left\{\int_0^1 x\ dx\right\}\left\{\int_0^1 (1-y)\ dy\right\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

(iii) We have $A = \{(x,y)\ :\ 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0,y)$, and then integrate dy on the range $(0,1)$, that is

$$\begin{aligned} \mathrm{P}[\ X < Y\ ] &= \int_A\int f_{X,Y}(x,y)\ dxdy = \int_0^1 \left\{\int_0^y 4x(1-y)\ dx\right\} dy = \int_0^1 \left\{\int_0^y x\ dx\right\} 4(1-y)\ dy \\ &= \int_0^1 2y^2(1-y)\ dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6} \end{aligned}$$

10. The joint pdf of X and Y is given by

$$f_{X.Y}(x,y) = 24xy \qquad x>0,\ y>0,\ x+y<1$$

and zero otherwise, the marginal pdf f_X is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} \\ &= 12x(1-x)^2 \qquad\qquad 0<x<1 \end{aligned}$$

as the integrand is only non-zero when $0 < x + y < 1 \Longrightarrow 0 < y < 1 - x$ for fixed x

11. (a) We have

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left\{F_1(t_1,t_2)\right\}_{t_1=x,t_2=y} = -e^{-x-y} < 0$$

on the specified range of X and Y, so $F_{X,Y}$ is not a valid cdf, as this partial derivative must be non-negative.

(b) We have

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left\{F_2(t_1,t_2)\right\}_{t_1=x,t_2=y} = \begin{cases} e^{-y} & 0 \le x \le y \\ \\ e^{-x} & 0 \le y \le x \end{cases}$$

which is non-negative everywhere. Note that, $F_2(x,0) = F_2(0,y) = 0$. However, consider the behaviour of F_2 as x and y become large; first, consider the cdf F defined by

$$F(x,y) = 1 - e^{-y} - ye^{-x} \qquad 0 \le y \le x < \infty,$$

that is, identical to F_2 on only half the original domain. It is easy to check that F is a cdf, in particular, that

$$\lim_{x,y\longrightarrow\infty} F(x,y) = 1$$

Hence, by symmetry we must have that

$$\lim_{x,y\longrightarrow\infty} F_2(x,y) = 2,$$

so F_2 is not a valid cdf.

Changing the question slightly gives a different solution; if cdf F_2 is defined as

$$F_2(x,y) = \begin{cases} 1 - e^{-x} - xe^{-y} & 0 \le x \le y \\ \\ 1 - e^{-y} - ye^{-y} & 0 \le y \le x \end{cases}$$

then we have

$$f_2(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\left\{F_2(t_1,t_2)\right\}_{t_1=x,t_2=y} = e^{-y} \qquad 0 \le x \le y < \infty$$

and zero otherwise, which is also non-negative everywhere. Again, $F_2(x,0) = F_2(0,y) = 0$, and here

$$\lim_{x,y\longrightarrow\infty} F_2(x,y) = 1$$

so F_2 **is** a valid cdf.

Note also that, in the amended question the marginal cdfs for X and Y are given by

$$\begin{aligned} F_X(x) &= \lim_{y\longrightarrow\infty} F_2(x,y) = 1 - e^{-x} \qquad x \ge 0 \\ F_Y(y) &= \lim_{x\longrightarrow\infty} F_2(x,y) = 1 - (y+1)e^{-y} \qquad y \ge 0 \end{aligned}$$

12. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$. The marginals are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_{1/x}^{x} \frac{1}{2x^2 y}\ dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\ dx = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\ dx = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

Marginal expectation of Y;

$$\mathrm{E}_{f_Y}[\,Y\,] = \int_{-\infty}^{\infty} y f_Y(y)\ dy = \int_0^1 \frac{y}{2}\ dy + \int_1^{\infty} \frac{1}{2y}\ dy = \infty$$

as the second integral is divergent.

13. To compute c,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dxdydz = \int_0^1\int_0^z\int_0^y c\ dxdydz = c\int_0^1\int_0^z y\ dydz = c\int_0^1 z^2/2\ dz = c/6$$

so $c = 6$.

$$f_{X,Z}(x,z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy = \int_x^z 6\ dy = 6(z-x) \qquad 0 < x < z < 1$$

$$f_{Y,Z}(y,z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dx = \int_0^y 6\ dx = 6y \qquad 0 < y < z < 1$$

$$f_{Y|X,Z}(y|x,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Z}(x,z)} = \frac{1}{z-x} \qquad x < y < z$$

$$f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} = \frac{1}{y} \qquad 0 < x < y$$

$$f_{X,Y|Z}(x,y|z) = \frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)} = \frac{2}{z^2} \qquad x < y < z$$

as

$$f_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy\ dx = \int_0^z\int_0^y 6\ dxdy = 3z^2$$