M2S1 : ASSESSED COURSEWORK 2 : SOLUTIONS

(a) We have

$$f_{X,Y}(x,y) = c_1(x+y) \qquad\qquad 0 \le x, y \le 1$$

and zero otherwise.

(i) To compute c_1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy = 1$$

and as

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy &= \int_0^1\int_0^1 f_{X,Y}(x,y)\,dxdy = \int_0^1\int_0^1 c_1(x+y)\,dxdy \\ &= c_1\int_0^1 \left[\frac{x^2}{2}+xy\right]_0^1 dy = c_1\int_0^1\left(\frac{1}{2}+x\right)dy = c_1\left[\frac{x}{2}+\frac{x^2}{2}\right]_0^1 = c_1\end{aligned}$$

so that $c_1 = 1$.

[1 MARK]

(ii) To compute the marginal for X, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \qquad 0 \le x \le 1$$

[2 MARKS]

Note that, by symmetry, we also have

$$f_Y(y) = y + \frac{1}{2} \qquad 0 \le y \le 1$$

(iii) Using the result from (ii),

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt = \int_0^y \left(t+\frac{1}{2}\right)dt = \left[\frac{t^2}{2}+\frac{t}{2}\right]_0^y = \frac{y^2}{2}+\frac{y}{2} = \frac{y}{2}(1+y) \qquad 0 \le y \le 1$$

so that

$$P\left[Y < \frac{1}{2}\right] = F_Y\left(\frac{1}{2}\right) = \frac{3}{8}$$

[2 MARKS]

(iv) Can write

$$P\left[Y < X^2\right] = \int\int_A f_{X,Y}(x,y)dxdy$$

where A is the set

$$A \equiv \left\{(x,y) : y < x^2\right\}$$

so that, integrating dx first (for fixed y, on the range from $\left(\sqrt{y}, 1\right)$)

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_0^1 \left\{\int_{\sqrt{y}}^1 (x+y)dx\right\} dy = \int_0^1 \left\{\left[\frac{x^2}{2} + xy\right]_{\sqrt{y}}^1\right\} dy \\
&= \int_0^1 \left\{\frac{1}{2} + y - \frac{y}{2} - \sqrt[3]{y}\right\} dy = \int_0^1 \left\{\frac{1}{2} + \frac{y}{2} - \sqrt[3]{y}\right\} dy \\
&= \left[\frac{y}{2} + \frac{y^2}{4} - \frac{2}{5}\sqrt[5]{y}\right]_0^1 = \frac{1}{2} + \frac{1}{4} - \frac{2}{5} = \frac{7}{20}
\end{aligned}$$

or, equivalently, integrating dy first (for fixed x, on the range from $\left(0, x^2\right)$)

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_0^1 \left\{\int_0^{x^2} (x+y)dy\right\} dx = \int_0^1 \left\{\left[xy + \frac{y^2}{2}\right]_0^{x^2}\right\} dx = \int_0^1 \left\{x^3 + \frac{x^4}{2}\right\} dx \\
&= \left[\frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}
\end{aligned}$$

[3 MARKS]

(iv) Under the new specification, a change in the range of integration changes the normalization constant from c_1 to c_2 and also the value $P\left[Y < X^2\right]$. First, to compute c_2, we integrate dy for fixed x; for any fixed x the range of integration dy is from the axis to the line $x+y=1$, that is, from $y=0$ to $y=1-x$ and hence

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= \int_0^1 \left\{\int_0^{1-x} f_{X,Y}(x,y)dy\right\} dx = \int_0^1 \left\{\int_0^{1-x} c_2(x+y)dy\right\} dx \\
&= c_2 \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx \\
&= c_2 \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx = \frac{c_2}{2}\int_0^1 \left(2x(1-x)+(1-x)^2\right) dx \\
&= \frac{c_2}{2}\int_0^1 \left(1-x^2\right) dx \\
&= \frac{c_2}{2}\left[x - \tfrac{x^3}{3}\right]_0^1 = \frac{c_2}{3}
\end{aligned}$$

so that $c_2 = 3$.

Now, to calculate $P\left[Y < X^2\right]$ first note that the region A of interest is the region bounded by the horizontal axis and the lines $x+y=1$ and $y=x^2$. These lines meet when

$$x^2 = 1-x \therefore x^2+x-1=0 \iff x = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

and we require the root that lies in the range of X, that is

$$\frac{-1+\sqrt{5}}{2} = \alpha \qquad \text{say.}$$

Now, by insepction of a suitable sketch, we see that again integration dy for fixed x, we need to split the range of x into two ranges, that is, first, from 0 to α and second from α to , as the range of integration dy is **different** in these two cases.

$$\begin{aligned}
P\left[Y < X^2\right] &= \int_A \int f_{X,Y}(x,y)dydx = \int_0^{\alpha} \left\{ \int_0^{x^2} 3(x+y)dy \right\} dx + \int_{\alpha}^{1} \left\{ \int_0^{1-x} 3(x+y)dy \right\} dx \\
&= 3\int_0^{\alpha} \left\{ \left[xy + \frac{y^2}{2} \right]_0^{x^2} \right\} dx + 3\int_{\alpha}^{1} \left\{ \left[xy + \frac{y^2}{2} \right]_0^{1-x} \right\} dx \\
&= 3\int_0^{\alpha} \left\{ x^3 + \frac{x^4}{2} \right\} dx + 3\int_{\alpha}^{1} \left\{ x(1-x) + \frac{(1-x)^2}{2} \right\} dx \\
&= 3\left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^{\alpha} + \frac{3}{2}\left[x - \frac{x^3}{3} \right]_{\alpha}^{1} = 3\left(\frac{\alpha^4}{4} + \frac{\alpha^5}{10} \right) + \frac{3}{2}\left(\frac{2}{3} - \left(\alpha - \frac{\alpha^3}{3} \right) \right) \\
&= \tfrac{3}{20}\left(5\alpha^4 + 2\alpha^5\right) + \left(1 - \frac{3}{2}\alpha + 2\alpha^3 \right) = \tfrac{1}{20}(6\alpha^5 + 15\alpha^4 + 10\alpha^3 - 30\alpha + 20)
\end{aligned}$$

which gives $P\left[Y < X^2\right] = 0.3274$.

[5 MARKS]

Here is a MAPLE check of the calculation : α

```
> evalf(int(int(3*(x+y),y=0..min(x^2,1-x)),x=0..1));
                              .3274575141
> a:=-1/2+sqrt(5)/2;
                         a := 1/2 sqrt(5) - 1/2
> alpha:=convert(a,float);
                            alpha := .6180339890
> (3/20)*(5*alpha^4+2*alpha^5)+(1-3*alpha/2+alpha^3/2);
                              .3274575144
```

(b) The multivariate transformation result for two variables is to be used. First, we have that

$$\left.\begin{array}{ll} R & =\sqrt{X^2+Y^2} \\ T & =\tan^{-1}\left(\dfrac{Y}{X}\right) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} X & =R\cos T \\ Y & =R\sin T \end{array}\right.$$

so that the range of R is $(0,\infty)$ and the range of T is $(0,2\pi)$, and the Jacobian is

$$J(r,t)=\left|\begin{array}{cc} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial t} \end{array}\right| = \left|\begin{array}{cc} \cos t & -r\sin t \\ \sin t & r\cos t \end{array}\right| = r\sin^2 t + r\cos^2 t = r$$

Hence, by the transformation result

$$f_{R,T}(r,t)=f_{X,Y}(r\cos t, r\sin t)\,J(r,t)=\frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(r^2\cos^2 t+r^2\sin^2 t\right)\right\}r=\frac{1}{2\pi}r\exp\left\{-\frac{r^2}{2}\right\}$$

for $0<r$, $0<t<2\pi$, and zero otherwise. Now, the marginal pdf for R, f_R is given by

$$f_R(r)=\int_{-\infty}^{\infty} f_{R,T}(r,t)dt=\int_0^{2\pi}\frac{1}{2\pi}r\exp\left\{-\frac{r^2}{2}\right\}dt=\frac{1}{2\pi}r\exp\left\{-\frac{r^2}{2}\right\}\int_0^{2\pi}dt=r\exp\left\{-\frac{r^2}{2}\right\}$$

for $0<r$ and zero otherwise. Similarly the marginal pdf for T, f_T is given by

$$f_T(t)=\int_{-\infty}^{\infty} f_{R,T}(r,t)dr=\int_0^{\infty}\frac{1}{2\pi}r\exp\left\{-\frac{r^2}{2}\right\}dr=\frac{1}{2\pi}\int_0^{2\pi}r\exp\left\{-\frac{r^2}{2}\right\}dr=\frac{1}{2\pi}\left[\exp\left\{-\frac{r^2}{2}\right\}\right]_0^{\infty}=\frac{1}{2\pi}$$

for $0<t<2\pi$, and zero otherwise. Hence

$$f_{R,T}(r,t)=f_R(r)f_T(t)$$

and the joint range is a Cartesian product of the range of each variable, and so R and T are independent random variables.

[*5 MARKS*]

(ii) The joint pdf of variables U and V is obtained as follows. First, note

$$\left.\begin{array}{ll} U & =\exp\left\{-\frac{R^2}{2}\right\} \\ V & =\dfrac{T}{2\pi} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} R & =\sqrt{-2\log U} \\ T & =2\pi V \end{array}\right.$$

so that the range of U and V is $(0,1)$ and the Jacobian is

$$J(u,v)=\left|\begin{array}{cc} \dfrac{\partial r}{\partial u} & \dfrac{\partial r}{\partial v} \\ \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial v} \end{array}\right| = \left|\begin{array}{cc} \dfrac{-1}{U\sqrt{-2\log U}} & 0 \\ 0 & 2\pi \end{array}\right| = \frac{2\pi}{U\sqrt{-2\log U}}$$

Hence, by the transformation result

$$f_{U,V}(u,v) = f_{R,T}\left(\sqrt{-2\log u}, 2\pi v\right) J\left(u,v\right) = f_R\left(\sqrt{-2\log u}\right) f_T\left(2\pi v\right) J\left(u,v\right) \qquad 0 < u,v < 1$$

which gives

$$f_{U,V}(u,v) = \sqrt{-2\log u}\exp\left\{-\frac{-2\log U}{2}\right\}\cdot\frac{1}{2\pi}\cdot\frac{2\pi}{u\sqrt{-2\log u}} = 1 \qquad 0 < u,v < 1$$

and hence

$$f_{U,V}(u,v) = f_U(u)f_V(v) \qquad 0 < u,v < 1$$

and U and V are independent (and each is $Uniform(0,1)$).

[2 MARKS]

Note that this full calculation can be circumvented by noting that U is a function of R only, and V is a function of T only, and hence as R and T are independent then U and V are also automatically independent. Note also that the transformations from R to U and the transformation from T to V are both transformations via functions that are the cdfs of the original variable, that is,

$$U = F_R(R) \qquad\qquad V = F_T(T)$$

and therefore, by the result given in lectures, we must have that U and V are $Uniform(0,1)$)