M2S1 : ASSESSED COURSEWORK 1 : SOLUTIONS

1.(i) Using the formula given in the question for f_S

$$f_S(s) = \sum_{x=0}^{s} f_X(x) f_Y(s-x)$$

we have for the pgf

$$\begin{aligned}
G_S(t) &= \sum_{s=0}^{\infty} t^s f_S(s) = \sum_{s=0}^{\infty} t^s \left\{ \sum_{x=0}^{s} f_X(x) f_Y(s-x) \right\} = \sum_{s=0}^{\infty} \sum_{x=0}^{s} t^s f_X(x) f_Y(s-x) \\
&= \sum_{x=0}^{\infty} \sum_{s=x}^{\infty} t^s f_X(x) f_Y(s-x) \\
&= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} t^{x+y} f_X(x) f_Y(y) \qquad \text{(setting } y = s-x\text{)} \\
&= \left\{ \sum_{x=0}^{\infty} t^x f_X(x) \right\} \left\{ \sum_{y=0}^{\infty} t^y f_Y(y) \right\} = G_X(t) G_Y(t)
\end{aligned}$$

[*5 MARKS*]

(ii) For the pmf given (which is Binomial with parameters n and θ)

$$\begin{aligned}
G_{S_i}(t) &= \sum_{x=0}^{\infty} t^x f_{S_i}(x) = \sum_{x=0}^{n} t^x \binom{n}{x} \theta^x (1-\theta)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (\theta t)^x (1-\theta)^{n-x} \\
&= (1-\theta+\theta t)^n
\end{aligned}$$

by the binomial theorem.

[*3 MARKS*]

(iii) **Given** that $R = r$, we seek the pmf of the sum

$$S = \sum_{i=1}^{r} S_i$$

which we can compute using the a key pgf result from lectures which is the extension of the result in (i), namely that as the S_i variables are independent we have for the conditional pgf

$$G_S(t) = \prod_{i=1}^{r} G_{S_i}(t)$$

so that from (ii),

$$G_S(t) = \prod_{i=1}^{r} (1-\theta+\theta t)^n = (1-\theta+\theta t)^{nr}$$

which is a pgf of a form identical to that in (ii) but with a different power. Hence this is the pgf coresponding to the pmf

$$\binom{nr}{x} \theta^x (1-\theta)^{nr-x} \qquad x \in \{0, 1, ..., nr\}$$

Hence if the conditional pmf given $R = r$ is denoted $f_{S|R}(s|r)$ then

$$f_{S|R}(s|r) = \mathrm{P}\left[S = s|R = r\right] = \binom{nr}{s}\theta^s\left(1-\theta\right)^{nr-s} \qquad s \in \{0, 1, ..., nr\}$$

[*4 MARKS*]

(iv) Finally, the unconditional pmf of S, $f_S(s)$, is obtained from (iii) using the Theorem of Total Probability, as

$$\begin{aligned} f_S(s) &= \mathrm{P}\left[S = s\right] = \sum_{r=0}^{\infty} \mathrm{P}\left[S = s|R = r\right]\mathrm{P}\left[R = r\right] \\ &= \sum_{r=0}^{\infty}\binom{nr}{s}\theta^s\left(1-\theta\right)^{nr-s}\frac{e^{-\lambda}\lambda^r}{r!} \qquad s = 0, 1, 2, ... \end{aligned}$$

which is an expression that cannot be simplified usefully. However, if we try to compute the (unconditional) pgf of S then

$$\begin{aligned} G_S(t) &= \sum_{s=0}^{\infty} t^s f_S(s) = \sum_{s=0}^{\infty} t^s \left\{\sum_{r=0}^{\infty}\mathrm{P}\left[S = s|R = r\right]\mathrm{P}\left[R = r\right]\right\} \\ &= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} t^s\mathrm{P}\left[S = s|R = r\right]\mathrm{P}\left[R = r\right] \\ &= \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{nr} t^s\mathrm{P}\left[S = s|R = r\right]\right\}\mathrm{P}\left[R = r\right] && \text{exchanging order of summation, noting } \mathrm{P}\left[S = s|R = r\right] = 0 \text{ if } s > nr \\ &= \sum_{r=0}^{\infty}\left\{\sum_{s=0}^{nr} t^s\binom{nr}{s}\theta^s\left(1-\theta\right)^{nr-s}\right\}\mathrm{P}\left[R = r\right] \\ &= \sum_{r=0}^{\infty}\left\{\left(1-\theta+\theta t\right)^{nr}\right\}\mathrm{P}\left[R = r\right] && \text{from (ii)} \\ &= \sum_{r=0}^{\infty}\left(1-\theta+\theta t\right)^{nr}\frac{e^{-\lambda}\lambda^r}{r!} = e^{-\lambda}\sum_{r=0}^{\infty}\frac{\left(\lambda\left(1-\theta+\theta t\right)^n\right)^r}{r!} \\ &= e^{-\lambda}\exp\left\{\lambda\left(1-\theta+\theta t\right)^n\right\} && \text{as the sum is that of an exponential type.} \\ &= \exp\left\{-\lambda(1-\left(1-\theta+\theta t\right)^n\right\} \end{aligned}$$

[*8 MARKS*]