# M2S1 PROBABILITY AND STATISTICS

## SAMPLE EXAM QUESTIONS : II

1. (a) For each of the following infinite sequences of events $E_1, E_2, ...$, determine whether the sequence is Non-Increasing, Non-Decreasing, or not monotone. If the sequence is monotone, find and interpret in words the limit event. If the sequence is not monotone, find and interpret in words the limsup amd liminf events, and state whether the sequence is convergent.

(i) A coin is repeatedly tossed. The event $E_k$ occurs if the $k$th toss is a Head.

(ii) A coin is repeatedly tossed. The event $E_k$ occurs if the first $k + 10$ tosses contain a run of 10 Heads.

(iii) In a measurement experiment, the event $E_k$ occurs if the measurement lies in the (open) interval $(a + \alpha_k, b + \beta_k)$, where $\alpha_1, \alpha_2, ...$ and $\beta_1, \beta_2, ...$ are monotone decreasing sequences of positive real constants, with

$$\alpha_k, \beta_k \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

(b) Suppose that $E_1, E_2, ...$ is a non-decreasing sequence of events in event sample space $\Omega$.

(i) Show that the sequence of events $F_1, F_2, ...$ defined by

$$F_k = E_k^{'} \qquad k = 1, 2, ...$$

is non-increasing

(ii) Show that if $E$ and $F$ are the limit events corresponding to to $E_1, E_2, ...$ and $F_1, F_2, ...$ respectively, then

$$E \cup F = \Omega$$

[Hint: think about the limit events in terms of sample outcomes.]

(iii) Assuming that, for a non-increasing sequence of events $F_1, F_2, ...$,

$$\mathrm{P}(F) = \mathrm{P}\left(\lim_{k \to \infty} F_k\right) = \lim_{k \to \infty} \mathrm{P}(F_k),$$

show that the result also holds for a non-decreasing sequence.

2. (a) Suppose that continuous random variable $X$ takes values on the interval $(0, 2)$ with probability density function (pdf) $f_X$ defined by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

(i) Sketch $f_X$

(ii) Find and sketch the cumulative distribution function of $X$, $F_X$.

(iii) Find P[ $0.2 \leq X \leq 1.2$ ]

(iv) Find P[ $X > 1.5 \mid X > 1$ ]

(b) Suppose that continuous random variables $Y$ and $Z$ have joint pdf $f_{Y,Z}$ given by

$$f_{Y,Z}(y, z) \;=\; c \qquad 0 \leq y \leq 2,\; 0 \leq z \leq 1,\; 2z \leq y,$$

and zero otherwise, for some constant $c$, that is, constant on some region $\mathcal{A} \subset \mathcal{R}^{\in}$.

(i) Sketch the region $\mathcal{A}$ on which $f_{Y,Z}$ is non-zero.

(ii) Find the value of $c$.

(iii) Find P[$Y > 3Z$]

(iv) Find the marginal pdfs of $Y$ and $Z$.

3. (a) Suppose that $X$ and $Y$ are independent continuous random variables each having a standard (unit) Normal distribution.

(i) Show that random variable $Z$, defined by $Z = X - Y$, has a Normal distribution.

(ii)Find the distribution of random variable $S$, defined by

$$S = \frac{Z^2}{2}$$

(iii) Find the distribution of random variable $U$ defined by

$$U = |Y|$$

(b)(i) Find the distribution of random variable $V$ defined by

$$V = \frac{X}{U}$$

(ii) Find the distribution of random variable $W$ defined by

$$W = \frac{X}{X+Y}$$

4. (a) Let $X$ and $Y$ be independent continuous random variables taking values on the whole real line with cumulative distributions $F_X$ and $F_Y$ respectively.

(i) Define continuous random variable $Z$ by

$$Z = \mathrm{Min}(X, Y).$$

Find the cdf of $Z$, $F_Z$, in terms of $F_X$ and $F_Y$.

(ii) Define discrete random variable $W$ in terms of $X$ and $Y$ by

$$W = 1 \text{ if } X < Y$$

$$W = 0 \text{ if } X \geq Y.$$

Show that

$$\mathrm{P}[W = 1] = \int_{-\infty}^{\infty} (1 - F_Y(x))) f_X(x) dx$$

(iii) Show that the joint probability of the events $[Z \leq z]$ and $[W = 1]$ can be written

$$\mathrm{P}[Z \leq z, W = 1] = \mathrm{P}[X \leq z, X < Y]$$

for any real $z$.

(b) Show that

$$\mathrm{P}[Z \leq z, W = 1] = \int_{-\infty}^{z} F_X(y) f_Y(y) dy \; + \; F_X(z)\,(1 - F_Y(z))$$

and hence show (by differentiation) that the joint density of $Z$ and $W$ evaluated at $w = 1$ is given by

$$f_{Z,W}(z, w) = f_X(z)\,(1 - F_Y(z)) \quad - \infty < z < \infty, w = 1$$

Find a similar expression for the joint density evaluated at $w = 0$ (by considering the joint probability $\mathrm{P}[Z \leq z, W = 0]$, or otherwise).

[Hint: use the result

$$\mathrm{P}[X \leq z, X < Y] = \int_{-\infty}^{\infty} \mathrm{P}[X \leq z, X < y] f_Y(y) dy$$

and consider the integral carefully.]

5. (a) Suppose that $Y$ is a discrete random variable having a Poisson distribution with parameter $\lambda$, and suppose that $X$ is a continuous random variable that, conditional on $Y = y$, has a Chi-squared distribution with $2y$ degrees of freedom, that is,

$$Y \sim Poisson(\lambda) \qquad X|Y = y \sim \chi^2_{2y}$$

Find the (marginal) expectation and variance of $X$.

(b) Show that the moment generating function of $X$, $M_X$, is given by

$$M_X(t) = \exp\left\{\frac{2\lambda t}{1-2t}\right\}$$

and hence verify the results obtained in (a).

In answering (a) and (b), you may use the following results;

If $Z \sim Poisson(\lambda)$, then $E_{f_Z}[Z] = \lambda$, $E_{f_Z}[Z^2] = \lambda(\lambda + 1)$

If $Z \sim Ga(\alpha, \beta)$, then $E_{f_Z}[Z] = \alpha/\beta$, $E_{f_Z}[Z^2] = \alpha(\alpha + 1)/\beta^2$, and the mgf of $Z$ is given by

$$M_Z(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$$

6.(a) Suppose that $X_1, ..., X_n$ are independent and identically distributed continuous random variables each having a $Uniform(0,1)$ distribution. Let $Y_1 = \mathrm{Min}(X_1, ..., X_n)$ and $Y_n = \mathrm{Max}(X_1, ..., X_n)$ be the corresponding minimum and maximum order statistics, respectively

(i)Show that the joint probability of the events $[Y_1 > y_1]$ and $[Y_n \leq y_n]$ can be expressed

$$\mathrm{P}[Y_1 > y_1, Y_n \leq y_n] = (y_n - y_1)^n \quad 0 < y_1 < y_n < 1$$

and zero otherwise.

(ii) Show also that

$$\mathrm{P}[Y_1 > y_1, Y_n \leq y_n] = \mathrm{P}[Y_n \leq y_n] - \mathrm{P}[Y_1 \leq y_1, Y_n \leq y_n]$$

Hence (or otherwise) derive the joint density function of $Y_1$ and $Y_n$.

(b) Find an expression for the marginal density of random variable $Z$ defined by

$$Z = \frac{Y_1}{Y_n},$$

and show that $Z$ and $Y_n$ are independent.

7. (a) Let $X$ be a discrete random variable taking values in the set $\{1, c\}$ for some real constant $c > 0$, with mass function $f_X$ defined by

$$f_X(x) = \begin{cases} \frac{1}{2} & x = 1 \\ \\ \frac{1}{2} & x = c \end{cases}$$

(i) Calculate the expectation and variance of $X$.

(ii) Calculate the expectation and variance of random variable $Z = \log X$.

(b)(i) Suppose that $X_1, ..., X_n$ are independent and identically distributed random variables having the same distribution as $X$. The *geometric* mean of $X_1, ..., X_n$, $Y_n$, is a continuous random variable defined by

$$Y_n = \left( \prod_{i=1}^{n} X_i \right)^{1/n}$$

Using Jensen's inequality, show that

$$E_{f_{Y_n}}[Y_n] \geq \sqrt{c}$$

[Hint: use $g(x) = \exp\{x\}$].

(ii) Using the Central Limit Theorem, or otherwise, find the distribution of the random variable $S_n$ defined by

$$S_n = \sqrt{n}(Y_n - \sqrt{c})$$

as $n \longrightarrow \infty$, and hence show that $Y_n$ converges in probability to $\sqrt{c}$,

$$Y_n \xrightarrow{P} \sqrt{c}$$

**JENSEN'S INEQUALITY**

If $X$ is a random variable, and function $g$ is convex, then

$$\mathrm{E}_{f_X}[g(X)] \geq g(\mathrm{E}_{f_X}[X])$$

8.(a) Suppose that $X_1, ..., X_n$ are independent and identically distributed Exponential random variables with parameter $\lambda = 1$.

(i) State the distribution of random variable $S_n$ defined by $S_n = X_1 + ... + X_n$.

(ii) Let random variable $Y_n$ be defined by

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

that is, the sample mean of $X_1, ..., X_n$.

Show that the pdf of $Y_n$, $f_{Y_n}$ is given by

$$f_{Y_n}(y) = \frac{n^n}{\Gamma(n)} y^{n-1} \exp\{-ny\} \qquad y > 0.$$

Hence find the pdf of random variable $Z_n$ defined by

$$Z_n = \frac{Y_n - 1}{1/\sqrt{n}}$$

(b) Find an approximation to the cdf of $Z_n$ for large $n$, and hence verify Stirling's formula,

$$\Gamma(n) \approx \sqrt{2\pi} n^{n-1/2} \exp\{-n\}$$

9. (a) Suppose that $X_1, ..., X_n$ are independent and identically distributed continuous random variables having pdf $f_X$ given by

$$f_X(x) = (\theta + 1)x^\theta \qquad 0 < x < 1$$

and zero elsewhere for some $\theta > 0$.

(i) Find the expectation of $X_1, ..., X_n$

(ii) Find the form of the method of moments estimator of $\theta$.

(iii) Find the form of the maximum likelihood estimator of $\theta$.

(b) Suppose now that $X_1, ..., X_n$ are independent and identically distributed continuous random variables having pdf $f_X$ given by

$$f_X(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} \qquad \beta < x$$

for some $\alpha, \beta > 0$.

(i) Find the form of the maximum likelihood estimators of $\alpha$ and $\beta$.

(ii) Let $V_n$ be a random variable having the same form as the maximum likelihood estimator of $\beta$. Find the limiting distribution of $V_n$ as $n \longrightarrow \infty$.

10. (a) Suppose that $X_1, ..., X_n$ are independent and identically distributed random variables having a continuous uniform distribution on the interval $(0, \theta)$ for some $\theta > 0$.

(i) Find the form of the maximum likelihood estimator of $\theta$.

(ii) Let $Y_n$ be a random variable having the same form as the maximum likelihood estimator of $\theta$. Show that the pdf of $Y_n$, $f_{Y_n}$ is of the form

$$f_{Y_n}(y) = \frac{ny^{n-1}}{\theta^n} \qquad 0 < y < \theta,$$

and zero otherwise.

Show also that

$$\mathrm{E}_{f_{Y_n}}[Y_n] = \frac{n}{n+1}\theta,$$

and hence find an unbiased estimator of $\theta$.

(b)(i) Show that the statistic, $Z_n$, defined by

$$Z_n = \frac{2}{n}\sum_{i=1}^{n} X_i$$

is an unbiased estimator of $\theta$

(ii) Find the variances of $Y_n$ and $Z_n$.