If X has probability generating function $G_X(t) = \exp\{\lambda(t-1)\}$, then to find the mass function, we expand G_X and identify the coefficient of the term t^x

$$
\begin{aligned}
G_X(t) &= \exp\{\lambda(t-1)\} \\
&= \sum_{i=0}^{\infty} \frac{\lambda^i (t-1)^i}{i!} && \text{exponential power series} \\
&= \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \left\{ \sum_{j=0}^{i} \binom{i}{j} t^j (-1)^{i-j} \right\} && \text{binomial expansion of } (t-1)^i \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{1}{i!} \frac{i!}{j!(i-j)!} \lambda^i t^j (-1)^{i-j} \\
&= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{j!(i-j)!} \lambda^i t^j (-1)^{i-j} && \text{exchanging the order of summation} \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{i=j}^{\infty} \frac{1}{(i-j)!} \lambda^{i-j} (-1)^{i-j} \right\} \frac{\lambda^j t^j}{j!} \\
& \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!} \right\} \frac{\lambda^j}{j!} t^j && \text{setting } k = i-j \\
& \sum_{j=0}^{\infty} \left\{ \frac{e^{-\lambda} \lambda^j}{j!} \right\} t^j
\end{aligned}
$$

and hence the coefficient of t^x is

$$
\frac{e^{-\lambda} \lambda^x}{x!}
$$

which is the Poisson mass function formula. The limit result

$$
X \sim Binomial(n, \theta) \to Poisson(\lambda) \qquad \text{as } n \to \infty, \theta \to 0 \text{ with } n\theta = \lambda \text{ constant}
$$

result allows us approximate binomial probability calculations by probability Poisson calculation. That is we have for n large and θ small

$$
Binomial(n, \theta) \approx Poisson(\lambda) \qquad \binom{n}{x} \theta^x (1-\theta)^{n-x} \approx \frac{e^{-n\theta}(n\theta)^x}{x!}
$$

POISSON PROCESS ASSUMPTIONS

THIS MATERIAL NOT EXAMINABLE

Let $X(t)$ be the random variable recording the number of events that occur in the interval $[0, t)$. Let

$$P_n(t) = \mathrm{P}\left[n \text{ events in the interval } [0, t)\right]$$

Let $O(\triangle t)$ denote a function of $\triangle t$ such that

$$\lim_{\triangle t \to 0} \frac{O(\triangle t)}{\triangle t} = 0$$

that is, $O(\triangle t)$ is negligible relative to $\triangle t$

For the Poisson process, we assume that

1 $X(0) = 0$

2 $\mathrm{P}[X(t+h) - X(t) = n \,|\, X(s) = m]$ $=\mathrm{P}[X(t+h) - X(t) = n]$ for all $0 \leq s \leq t$ and $h > 0$, that is, the numbers of events that occur in disjoint time intervals are **independent**

3 for some constant $\lambda > 0$

$$P\left[X(t + \triangle t) - X(t) = 1\right] = \lambda \triangle t + O(\triangle t)$$

that is, the probability of precisely one event occurring in a short time interval $[t, t + \triangle t)$ is proportional to $\triangle t$.

4

$$P\left[X(t + \triangle t) - X(t) \geq 2\right] = O(\triangle t)$$

that is, the probability of more than one event occurring in a short time interval $[t, t + \triangle t)$ is essentially zero.

Then we can show that

$$P_n(t) = \mathrm{P}\left[n \text{ events in the interval } [0, t)\right] = \mathrm{P}\left[X(t) = n\right] = \frac{e^{-\lambda t}\,(\lambda t)^n}{n!} \qquad n = 0, 1,$$

that is, $X(t) \sim Poisson(\lambda t)$.