1.8. COUNTING TECHNIQUES

Suppose that an experiment has N equally likely sample outcomes. If event E corresponds to a collection of sample outcomes of size $n(E)$, then

$$\mathrm{P}(E) = \frac{n(E)}{N}$$

so it is necessary to be able to evaluate $n(E)$ and N.

MULTIPLICATION THEOREM

If operations labelled $1, ..., r$ can be carried out in $n_1, n_2, ..., n_r$ ways respectively, then there are $n_1 \times n_2 \times ... \times n_r$ ways of carrying out the r operations in total.

PERMUTATIONS AND COMBINATIONS

DEFINITION

A **permutation** is an *ordered* arrangement of a set of items.

A **combination** is an *unordered* arrangement of a set of items.

SIMPLE COMBINATORIAL RESULTS

RESULT 1 The number of permutations of n items is $n! = n(n-1)...1$.

RESULT 2 The number of permutations of r from n items is

$$^nP_r = (n)_r = \frac{n!}{(n-r)!} = n(n-1)...(n-r+1) \qquad \text{(by the Multiplication Theorem).}$$

RESULT 3 The number of combinations of r from n items is

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (\text{as } ^nP_r = r!^nC_r).$$

RESULT 4 The number of permutations of n objects, comprising r_i items of type i for $i = 1, ..., k$ is equal to the number of ways of *partitioning* n objects into k subgroups with r_i in subgroup i, that is

$$\frac{n!}{r_1!r_2!...r_k!}.$$

COMBINATORIAL IDENTITIES: For $0 \leq r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r} \qquad \sum_{r=0}^{n} \binom{n}{r} = 2^n \qquad \binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

Binomial expansions: for $n > 0$,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \qquad (1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

SAMPLING FROM A FINITE POPULATION

A sample of size n is to be obtained from a finite population of N distinguishable objects. The sample obtained may be regarded as an *ordered* or *unordered* sequence, and can be obtained using *sampling with replacement* or *sampling without replacement*. The number of distinguishable samples that can be obtained under these experimental conditions are

	WITH REPLACEMENT	WITHOUT REPLACEMENT
ORDERED	N^n	$(N)_n$
UNORDERED	$\binom{N+n-1}{n}$	$\binom{N}{n}$

URN MODELS : Consider sampling n objects without replacement from a population of size N comprising R Type I objects and $N-R$ Type II objects. Then the probability that the sample contains r Type I objects is given by the *hypergeometric* formulae,

$$\frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}} \qquad \text{for Max}\{0, n-N+R\} \ \leq \ r \leq \text{Min}\{n, R\}$$

OCCUPANCY PROBLEMS: The general experiment involving allocating r balls to n (numbered) cells (with all allocations equally likely) has several special cases :

ALLOCATION PROBLEM	NUMBER OF WAYS
n balls to n cells with no cells empty	$\dfrac{n!}{n^n}$
r balls to n cells with no cell containing more than one ball	$\dfrac{(n)_r}{n^r}$
r balls to n cells with k balls in cell i, for $0 \leq k \leq r$	$\binom{r}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{r-k}$
distinguishable patterns of allocations of r balls to n cells	$\binom{n+r-1}{r}$
r balls to n cells with no cells empty (for $r \geq n$)	$\binom{r-1}{n-1}$

PROBABILITY CALCULATIONS

Recall that if an experiment has N equally likely sample outcomes, and event E corresponds to a collection of sample outcomes of size $n(E)$, then

$$\mathrm{P}(E) = \frac{n(E)}{N}$$

EXAMPLE 1. A True/False exam has 20 questions. Let E = "16 answers correct at random". Then

$$\mathrm{P}(E) = \frac{\text{Number of ways of getting 16 out of 20 correct}}{\text{Total number of ways of answering 20 questions}} = \frac{\binom{20}{16}}{2^{20}} = 0.0046$$

EXAMPLE 2. Sampling without replacement.

Consider an Urn Model with 10 Type I objects and 20 Type II objects, and an experiment involving sampling five objects without replacement. Let E = "precisely 2 Type I objects selected". We need to calculate N and $n(E)$ in order to calculate $\mathrm{P}(E)$. In this case N is the number of ways of choosing 5 from 30 items, and hence

$$N = \binom{30}{5}.$$

To calculate $n(E)$, we think of E occurring by first choosing 2 Type I objects from 10, and then choosing 3 Type II objects from 20, and hence, by the multiplication rule,

$$n(E) = \binom{10}{2}\binom{20}{3}. \qquad \Longrightarrow \qquad \mathrm{P}(E) = \frac{\binom{10}{2}\binom{20}{3}}{\binom{30}{5}} = 0.360$$

NOTE: This result can be obtained using a conditional probability argument; consider event $F \subseteq E$, where

$$F = \text{"sequence of objects 11222 obtained"} = \bigcap_{i=1}^{5} F_{ij}$$

where F_{ij} = "type j object obtained on draw i" $i = 1, ..., 5, j = 1, 2$. Then

$$\begin{aligned} \mathrm{P}(F) &= \mathrm{P}(F_{11})\mathrm{P}(F_{21}|F_{11})...\mathrm{P}(F_{52}|F_{11}, F_{21}, F_{32}, F_{42}) \\ &= \frac{10}{30}\frac{9}{29}\frac{20}{28}\frac{19}{27}\frac{18}{26} \end{aligned}$$

Now consider event G where

$$G = \text{"sequence of objects 12122 obtained"}$$

Then

$$\mathrm{P}(G) \;=\; \frac{10}{30}\frac{20}{29}\frac{9}{28}\frac{19}{27}\frac{18}{26}$$

i.e. $\mathrm{P}(G) = \mathrm{P}(F)$. In fact, **any** sequence containing two Type I and three Type II objects has this probability, and there are $\binom{5}{2}$ such sequences. Thus, as all such sequences are mutually exclusive,

$$\mathrm{P}(E) \;=\; \binom{5}{2}\frac{10}{30}\frac{9}{29}\frac{20}{28}\frac{19}{27}\frac{18}{26} \;=\; \frac{\binom{10}{2}\binom{20}{3}}{\binom{30}{5}}$$

as before.

EXAMPLE 3. Sampling with replacement.

Consider an Urn Model with 10 Type I objects and 20 Type II objects, and an experiment involving sampling five objects with replacement. Let $E =$ "precisely 2 Type I objects selected". Again, we need to calculate N and $n(E)$ in order to calculate $\mathrm{P}(E)$. In this case N is the number of ways of choosing 5 from 30 items with replacement, and hence $N \;=\; 30^5$.

To calculate $n(E)$, we think of E occurring by first choosing 2 Type I objects from 10, and 3 Type II objects from 20 in any order. Consider such sequences of selection

Sequence	Number of ways
11222	10.10.20.20.20
12122	10.20.10.20.20
.	.

etc., and thus a sequence with 2 Type I objects and 3 Type II objects can be obtained in $10^2 20^3$ ways. As before there are $\binom{5}{2}$ such sequences, and thus

$$\mathrm{P}(E) \;=\; \frac{\binom{5}{2}10^2 20^3}{30^5} \;=\; 0.329.$$

Again, this result can be obtained using a conditional probability argument; consider event $F \subseteq E$, where $F =$ "sequence of objects 11222 obtained". Then

$$\mathrm{P}(F) \;=\; \left(\frac{10}{30}\right)^2\left(\frac{20}{30}\right)^3$$

as the results of the draws are **independent**. This result is true for any sequence containing two Type I and three Type II objects, and there are $\binom{5}{2}$ such sequences that are mutually exclusive, so

$$\mathrm{P}(E) \;=\; \binom{5}{2}\left(\frac{10}{30}\right)^2\left(\frac{20}{30}\right)^3$$