2.3. EXPECTATIONS

For a **discrete** random variable X with range $\mathbb{X}$ with probability mass function f_X, the **expectation** or **expected value** of X with respect to f_X is defined by

$$\mathrm{E}_{f_X}[X] = \sum_{x=-\infty}^{\infty} x f_X(x) = \sum_{x \in \mathbb{X}} x f_X(x)$$

For a **continuous** random variable X with range $\mathbb{X}$ and pdf f_X, the **expectation** or **expected value** of X with respect to f_X is defined by

$$\mathrm{E}_{f_X}[X] = \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_{\mathbb{X}} x f_X(x)\ dx$$

NOTE: The expected value is a weighted average of the values of X, but the sum/integral may not be convergent, and hence the expected value may be infinite.

EXTENSION

Let g be a real-valued function whose domain includes $\mathbb{X}$. Then

$$\begin{aligned} \mathrm{E}_{f_X}[g(X)] &= \sum_{x=-\infty}^{\infty} g(x) f_X(x) \qquad \text{if X is discrete} \\ \mathrm{E}_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\ dx \quad \text{if X is continuous} \end{aligned}$$

EXAMPLE

Suppose that X is a continuous random variable with probability density function (pdf) given by

$$f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$$

and zero otherwise. Consider $g(x) = e^{-x}$. Then

$$\begin{aligned} \mathrm{E}_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\ dx = \int_{0}^{\infty} e^{-x} \lambda e^{-\lambda x}\ dx \qquad (f_X(x) = 0 \text{ for } x < 0) \\ &= \lambda \int_{0}^{\infty} e^{-(\lambda+1)x}\ dx = \frac{\lambda}{\lambda+1}. \end{aligned}$$

PROPERTIES OF EXPECTATIONS

Let X be a random variable with mass function/pdf f_X. Let g and h be real-valued functions whose domains include $\mathbb{X}$, and let a and b be constants. Then

$$\mathrm{E}_{f_X}[ag(X) + bh(X)] = a\mathrm{E}_{f_X}[g(X)] + b\mathrm{E}_{f_X}[h(X)]$$

as (in the continuous case)

$$\begin{aligned} \mathrm{E}_{f_X}[ag(X) + bh(X)] &= \int [ag(x) + bh(x)] f_X(x)\ dx \\ &= a \int g(x) f_X(x)\ dx + b \int h(x) f_X(x)\ dx \\ &= a\mathrm{E}_{f_X}[g(X)] + b\mathrm{E}_{f_X}[h(X)] \end{aligned}$$

SPECIAL CASES

(i) $\mathrm{E}_{f_X}[aX+b] = a\mathrm{E}_{f_X}[X] + b$

(ii) Consider $g(x) = (x - \mathrm{E}_{f_X}[X])^2$. Write $\mu = \mathrm{E}_{f_X}[X]$ (a constant that does not depend on x). Then

$$\begin{aligned}
\mathrm{E}_{f_X}[g(X)] &= \int (x-\mu)^2 f_X(x)\ dx = \int x^2 f_X(x)\ dx\ - 2\mu \int x f_X(x)\ dx + \mu^2 \int f_X(x)\ dx \\
&= \int x^2 f_X(x)\ dx - 2\mu^2 + \mu^2 = \int x^2 f_X(x)\ dx - \mu^2 = \mathrm{E}_{f_X}[X^2] - \{\mathrm{E}_{f_X}[X]\}^2 = \mathrm{Var}_{f_X}[X]
\end{aligned}$$

say. $\mathrm{Var}_{f_X}[X]$ is the **variance** and $\sqrt{\mathrm{Var}_{f_X}[X]}$ is the **standard deviation** of the distribution.

(iii) Consider $g(x) = x^k$ for $k = 1, 2, \ldots$. Then in the continous case

$$\mathrm{E}_{f_X}[g(X)] = \mathrm{E}_{f_X}[X^k] = \int x^k f_X(x)\ dx,$$

and $\mathrm{E}_{f_X}[X^k]$ is the kth **moment** of the distribution.

(iv) Consider $g(x) = (x-\mu)^k$ for $k = 1, 2, \ldots$. Then

$$\mathrm{E}_{f_X}[g(X)] = \mathrm{E}_{f_X}[(X-\mu)^k] = \int (x-\mu)^k f_X(x)\ dx,$$

and $\mathrm{E}_{f_X}[(X-\mu)^k]$ is the kth **central moment** of the distribution.

(v) Consider $g(x) = aX + b$. Then

$$\begin{aligned}
\mathrm{Var}_{f_X}[g(X)] &= \mathrm{E}_{f_X}[(aX+b-\mathrm{E}_{f_X}[aX+b])^2] = \mathrm{E}_{f_X}[(aX+b-a\mathrm{E}_{f_X}[X]-b)^2] \\
&= \mathrm{E}_{f_X}[(a^2(X-\mathrm{E}_{f_X}[X])^2] = a^2 \mathrm{Var}_{f_X}[X]
\end{aligned}$$

APPROXIMATION TO EXPECTATION AND VARIANCE

A Taylor series expansion method can be used to obtain approximations to expectations of functions of a random variable. Let X be a continuous random variable, with range $\mathbb{X}$ and pdf f_X. Suppose that the expectation and variance of X with respect to f_X are denoted μ and σ^2 respectively, and let g be a real-valued function whose domain includes $\mathbb{X}$. Then a Taylor approximation of g around μ is given for real-value x by,

$$g(x) \approx g(\mu) + (x-\mu)g^{'}(\mu) + \frac{1}{2}(x-\mu)^2 g^{''}(\mu)$$

where $g^{'}$ and $g^{''}$ are the first and second derivatives of g respectively.

Using the Taylor approximation, and ignoring terms in $(x-\mu)^k$ for $k = 3, 4, \ldots$, the expectation of $g(X)$ with respect to f_X is given approximately by

$$\mathrm{E}_{f_X}[g(X)] \approx g(\mu) + \frac{1}{2}\sigma^2 g^{''}(\mu).$$

and ignoring terms in $(x-\mu)^2$ and higher, the variance of $g(X)$ with respect to f_X is given approximately by

$$\mathrm{Var}_{f_X}[g(X)] \approx \sigma^2 \left\{ g^{'}(\mu) \right\}^2$$