# M2S1 - EXERCISES 6

*Estimators and their Properties*

1. Suppose that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Define statistics $T_1$ and $T_2$ by

$$T_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad T_2 = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Show that

$$E_{f_{T_1}}[T_1] = E_{f_{T_2}}[T_2] = \lambda.$$

2. Let $s_n^2$ denote the sample variance derived from a random sample of size $n$ from $N(\mu, \sigma^2)$, so that

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Show that

$$\frac{\sqrt{n-1}(s_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1) \qquad \text{so that} \qquad s_n^2 \xrightarrow{d} N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

3. Suppose that $X_1, ..., X_n$ are a random sample from a $Ga(\alpha, \beta)$ distribution. Find the method of moments estimators of $\alpha$ and $\beta$.

4. An estimator, $T$, is an *unbiased* estimator of function $\tau(\theta)$ of parameter $\theta$ if

$$\mathrm{E}_{f_T}[T] = \tau(\theta)$$

where $f_T$ is the *sampling distribution* of $T$. The *bias*, $b(T)$, and *Mean Squared Error*, **MSE**, of an estimator $T$ of $\tau(\theta)$ are defined respectively by

$$b(T) = \mathrm{E}_{f_T}[T] - \tau(\theta) \qquad\qquad MSE(T) = \mathrm{E}_{f_T}[(T - \tau(\theta))^2]$$

Suppose that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Find the maximum likelihood estimator of $\lambda$, and show that this estimator is unbiased. Also, find the maximum likelihood estimator of $\tau(\lambda) = e^{-\lambda}$ =P$[X = 0]$.

5. Find the maximum likelihood estimators of the the unknown parameters in the following probability densities on the basis of a random sample of size $n$.

(i) $f_X(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1, \theta > 0$.

(ii) $f_X(x;\theta) = (\theta+1)x^{-\theta-2}$, $1 < x, \theta > 0$.

(iii) $f_X(x;\theta) = \theta^2 x \exp\{-\theta x\}$, $0 < x, \theta > 0$.

(iv) $f_X(x;\theta) = 2\theta^2 x^{-3}$, $\theta \le x, \theta > 0$.

(v) $f_X(x;\theta) = \dfrac{\theta}{2}\exp\{-\theta|x|\}$, $-\infty < x < \infty$, $\theta > 0$.

(vi) $f_X(x;\theta_1,\theta_2) = \dfrac{1}{\theta_2 - \theta_1}$, $\theta_1 \le x \le \theta_2$.

(vii) $f_X(x;\theta_1,\theta_2) = \theta_1\theta_2^{\theta_1}x^{-\theta_1-1}$, $\theta_2 \le x$, $\theta_1, \theta_2 > 0$.

6. Suppose that $X_1, ..., X_n$ are a random sample from the probability distribution with pdf

$$f_X(x; \lambda, \eta) = \lambda e^{-\lambda(x-\eta)} \quad x > \eta$$

and zero otherwise. Find the maximum likelihood estimators of $\lambda$ and $\eta$.

7. Suppose that $X_1, ..., X_n$ are a random sample from the probability distribution with pdf

$$f_X(x; \theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x > 0.$$

Show that the sample mean $\bar{X}$ is an unbiased estimator of $\theta$. Show also that, if random variable $Y_1$ is defined as $Y_1 = \min\{X_1, ..., X_n\}$ then random variable $Z = nY_1$ is also unbiased for $\theta$.

8. Suppose that $X_1, ..., X_n$ are a random sample from a $Uniform(\theta - 1, \theta + 1)$ distribution. Show that the sample mean $\bar{X}$ is an unbiased estimator of $\theta$. Let $Y_1$ and $Y_n$ be the smallest and largest order statistics derived from $X_1, ..., X_n$. Show also that random variable $M = (Y_1 + Y_n)/2$ is an unbiased estimator of $\theta$.

9. An estimator $T$ for function $\tau(\theta)$ of $\theta$ is the *uniformly minimum variance unbiased estimator*, or **UMVUE**, of $\tau(\theta)$ if $T$ is unbiased for $\tau(\theta)$ and $T^*$is any other unbiased estimator of $\tau(\theta)$ then

$$\mathrm{Var}_{f_{T^*}}[T^*] \geq \mathrm{Var}_{f_T}[T] \qquad \textbf{for all } \theta \in \Theta$$

In the single parameter case, it can be shown that, if $T$ is an unbiased estimator of $\tau(\theta)$ based on a random sample of size $n$ from $f_X$, then

$$\mathrm{Var}_{f_T}[T] \geq \frac{[\tau'(\theta)]^2}{n\mathrm{E}_{f_X}\left[\left\{\frac{\partial}{\partial\theta}\{\log f_X(X;\theta)\}\right\}^2\right]} = \frac{[\tau'(\theta)]^2}{n\mathrm{E}_{f_X}\left[\{\partial(X;\theta)\}^2\right]} \qquad \text{, say}$$

where $\tau'(\theta)$ is the first partial derivative of $\tau(\theta)$ with respect to $\theta$.

This is the **Cramer-Rao Lower Bound** on the variance of an unbiased estimator. If an estimator can be found that has the minimum variance, then this estimator can be regarded as the "best" unbiased estimator.

Suppose that $X_1, ..., X_n$ are a random sample from the a $Geometric(\theta)$ distribution. Find the maximum likelihood estimator, $T$, of $\tau(\theta) = 1/\theta$. Show that $T$ is unbiased for $\tau(\theta)$, and show that, in fact, $T$ is the UMVUE of $\tau(\theta)$, by calculating the Cramer-Rao lower bound in this case, and showing that this bound is attained as the variance of $T$.

10. Suppose that $X_1, ..., X_n$ are a random sample from the a $N(0, \theta)$ distribution. Find the maximum likelihood estimator of $\theta$. Is this estimator unbiased, and if so, is it the UMVUE ?.

11. Suppose that $X_1, ..., X_n$ are a random sample from a $Ga(2, \lambda)$ distribution.

(i) Find the maximum likelihood estimator of $\lambda$.

(ii) Find the maximum likelihood estimator, denoted $T$ say, of $\tau = 1/\lambda$.

(iii) Find $\mathrm{E}_{f_T}[T]$ and $\mathrm{E}_{f_T}[T^2]$.

Prove that $T \xrightarrow{P} \tau$ as $n \longrightarrow \infty$.