M2S1 - EXERCISES 3

Moment Generating Functions / Discrete and Continuous Multivariate Distributions

1. *Transformations of Normal random variables.*
The continuous random variable Z with range $\mathbb{Z} = \mathbb{R}$ has pdf given by

$$f_z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} \qquad z \in \mathbb{R}$$

(a) Find the mgf of random variable Z, the pdf and the mgf of random variable X where

$$X = \mu + \frac{1}{\lambda}Z.$$

for parameters μ and $\lambda > 0$.

Find the expectation of X., and the expectation of the function $g(X)$ where $g(x) = e^x$.

(b) Suppose now Y is the random variable defined in terms of X by $Y = e^X$ Find the pdf of Y, and show that the expectation of Y is

$$\exp\left\{\mu + \frac{1}{2\lambda^2}\right\}$$

(c) Finally, let random variable T be defined by $T = Z^2$. Find the pdf and mgf of Z.

2. Suppose that random variable X has mgf given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}$$

Find the probability distribution, and the expectation and variance of X.

3. Suppose that random variable X has mgf given by

$$M_X(t) = \left(1 - \theta + \theta e^t\right)^n$$

for some θ, where $0 \leq \theta \leq 1$. Obtain a power series expansion for $M_X(t)$, and hence identify $\mathrm{E}_{f_X}[X^r]$ for $r = 1, 2, 3, ...$

4. Suppose that X is a continuous random variable with pdf

$$f_X(x) \quad = \quad \exp\{-(x+2)\} \qquad -2 < x < \infty$$

Find the mgf of X, and hence find the expectation and variance of X.

5. Suppose that X is a random variable with mass function/pdf f_X and mgf M_X. The *cumulant generating function* of X, K_X, is defined by $K_X(t) = \log\left[M_X(t)\right]$. Prove that

$$\frac{d}{dt}\left\{K_X(t)\right\}_{t=0} = \mathrm{E}_{f_X}[X] \qquad\qquad \frac{d^2}{dt^2}\left\{K_X(t)\right\}_{t=0} = \mathrm{Var}_{f_X}[X]$$

6. Suppose that X and Y are discrete random variables with joint mass function given by

$$f_{X,Y}(x,y) = \quad c\,\frac{2^{x+y}}{x!\; y!} \qquad x,\; y = 0, 1, 2, ...$$

and zero otherwise, for some constant c. Find the value of c, and the marginal mass functions of X and Y. Prove that X and Y are independent random variables.

7. Continuous random variables X and Y have joint cdf, $F_{X,Y}$ defined for $(x,y) \in \mathbb{R}^2$ by

$$F_{X,Y}(x,y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right) \qquad x \geq 0$$

and zero otherwise. Find the joint pdf, $f_{X,Y}$. Are X and Y independent ?

8. The joint pdf of continuous random variables X and Y is given by

$$f_{X,Y}(x,y) = cx^{\alpha-1} \qquad x > 0,\ 0 < y < \exp\{-\beta x^{\alpha}\}$$

for parameters $\alpha, \beta > 0$, and zero otherwise, for constant c. Find the marginal pdfs of X and Y.

9. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

and zero otherwise for some constant c. Are X and Y independent random variables ?

Find the value of c., and, for the set $A \equiv \{(x,y)\ :\ 0 < x < y < 1\}$, the probability

$$\mathrm{P}[\ X < Y\ ] = \iint_A f_{X,Y}(x,y)\ dxdy$$

10. Suppose that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise. Find the marginal pdf of X, f_X given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy$$

where, recall, the integrand is only non-zero for some part of the integral range.

Hint: Sketch the region on which the joint density is non-zero.

11. Are the following functions valid continuous joint cdfs for random variables X and Y ?

$$F_1(x,y) = 1 - e^{-x-y} \qquad x, y \geq 0 \qquad\qquad F_2(x,y) = \begin{cases} 1 - e^{-x} - xe^{-y} & 0 \leq x \leq y \\ \\ 1 - e^{-y} - ye^{-x} & 0 \leq y \leq x \end{cases}$$

12. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) \quad = \quad \frac{1}{2x^2 y} \qquad 1 \leq x < \infty,\ 1/x \leq y \leq x$$

and zero otherwise. Derive the marginal pdf of X, the marginal pdf of Y, the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$. Calculate the marginal expectation of Y, $E_{f_Y}[Y]$.

13. Suppose that X, Y and Z are continuous random variables with joint pdf given by

$$f_{X,Y,Z}(x,y,z) = \ c \qquad 0 < x < y < z < 1$$

and zero otherwise, for some constant c. Find the value of c, and

(a) the joint pdf of X and Z
(b) the joint pdf of Y and Z
(c) the conditional pdf of Y given $X = x$ and $Z = z$.
(d) the conditional pdf of X given $Y = y$ and $Z = z$.
(e) the joint conditional pdf of X and Y given $Z = z$.