M2S1 - ASSESSED COURSEWORK 3

To be handed in no later than Thursday, 13th December

1. (a) Suppose that $X_1, ..., X_n$ are i.i.d. random variables with cdf F_X. Find the limiting distribution, F_Z, of the sequence $\{Z_n\}$

(i) if $Z_n = \min\{X_1, ..., X_n\}$, and

$$F_X(x) = \frac{\exp\{x^2\} - 1}{\exp\{x^2\}} \qquad 0 \leq x < \infty$$

(ii) if $Z_n = \dfrac{\max\{X_1, ..., X_n\}}{n^{1/2}}$, and

$$F_X(x) = \frac{x^2 - 1}{x^2} \qquad 1 < x < \infty$$

[In each case, find the cdf of Z_n and take the limit as $n \to \infty$].

(b) Now, suppose that

$$F_X(x) = 1 - e^{-\lambda x} \qquad x > 0$$

so that $X_1, ..., X_n \sim \textit{Exponential}(\lambda)$, and let

$$U_n = \min\{X_1, ..., X_n\} \qquad V_n = \max\{X_1, ..., X_n\}$$

Find constants a_n and b_n such that if

$$Y_n = \frac{U_n}{a_n} \qquad \text{and} \qquad Z_n = V_n + b_n$$

then, as $n \to \infty$

$$F_{Y_n}(y) \to 1 - e^{-\lambda y} \qquad y > 0 \qquad \text{and} \qquad F_{Z_n}(z) \to \exp\left\{-e^{-\lambda z}\right\} \qquad z > 0$$

2. (a) Use the Central Limit Theorem to obtain an approximation, in terms of the unit normal cdf, $\Phi(.)$, to the probability

$$\mathrm{P}\left[T_n \leq c\right] = \mathrm{P}\left[\frac{T_n - n\mu}{\sqrt{n\sigma^2}} \leq \frac{c - n\mu}{\sqrt{n\sigma^2}}\right]$$

where $T_n = \sum_{i=1}^{n} X_i$, for a sequence of i.i.d random variables $X_1, ..., X_n$, with

$$\mathrm{E}_{f_X}[X] = \mu \qquad \text{and} \qquad \mathrm{Var}_{f_X}[X] = \sigma^2$$

when

(i) $n = 20,\ c = 12,\ X_i \sim Uniform(0,1)$.

(ii) $n = 30,\ c = 50,\ X_i \sim Gamma(4,2)$.

(b) It can be shown (using the convolution theorem) that if $X_1, X_2, ...$ are i.i.d *Cauchy* random variables, then the random variable

$$T_n = \sum_{i=1}^{n} X_i$$

has pdf given by

$$f_{T_n}(t) = \frac{n}{\pi\left(n^2 + t^2\right)} \qquad -\infty < t < \infty$$

(i) Find the pdf of the **sample mean** random variable

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

(ii) Find the limiting distribution of the sample mean random variable $\overline{X}$ as $n \to \infty$

(iii) Explain why the Central Limit Theorem **cannot** be used to construct an approximation to

$$P\left[T_n \leq c\right]$$

in this case.