M2S1 - ASSESSED COURSEWORK 1

To be handed in no later than Friday, 26th October

1. Consider discrete random variables X and Y with ranges $\{0,1,2,...\}$ and probability mass functions (pmfs) f_X and f_Y respectively. Define discrete random variable S by $S = X + Y$, and suppose that for all x, y

$$P[(X = x) \cap (Y = y)] = P[X = x, Y = y] = P[X = x]\, P[Y = y]$$

so that X and Y are *independent* random variables. It can be shown, using the Theorem of Total Probability, that the probability mass function of S is given by

$$f_S(s) = \sum_{x=0}^{s} f_X(x) f_Y(s-x) \qquad s = 0,1,2,...$$

(i) The *probability generating function (pgf)*, denoted G_Z, of a discrete random variable Z with probability mass function f_Z and range $\{0,1,2,...\}$ is defined uniquely for f_Z (when the sum is convergent) by

$$G_Z(t) = \sum_{z=0}^{\infty} t^z f_Z(z)$$

Show for the variables defined above that $G_S(t) = G_X(t)G_Y(t)$.

A study of the preferences of A-Level Mathematics students is to be carried out. In an attempt to minimize bias, a list of schools with A-Level Mathematics classes comprising precisely n students is identified, and each school is required to ask each student in the class.

“Which topic do you prefer: Statistics or Mechanics ?”

It is to be assumed that each student answers this question “Statistics” with probability θ $(0 < \theta < 1)$, and that students' answers are mutually independent. For school i that returns a complete response (that is, x “Statistics” answers and $n - x$ “Mechanics” answers, for some x) , the total number of “Statistics” answers is a random variable, S_i say, where the pmf of S_i is given by

$$f_{S_i}(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \qquad x \in \{0,1,2,...,n\}$$

(ii) Find the pgf of S_i.

The number of schools that provide a complete response to the survey is also a non-negative random variable, R say, where the probability distribution of R is well-modelled by the pmf

$$f_R(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x \in \{0,1,2,...\}$$

for some constant parameter $\lambda > 0$. Incomplete responses are discarded.

(iii) Using the results in (i) and (ii) and any results stated in lectures (or from M1S), and **given** that $R = r$, find the (conditional) pmf of the random variable S as defined as

$$S = \sum_{i=1}^{r} S_i$$

(iv) Find the (unconditional) pgf of S, G_S, if R is **not** observed.

M2S1 - SUPPLEMENTARY 1

(not to be handed in)

1. For which values of the constant c do the following functions define valid probability mass functions for a discrete random variable X, taking values on range $\mathbb{X} = \{1, 2, 3, ...\}$:

(a) $f_X(x) = c/2^x$ (b) $f_X(x) = c/(x2^x)$

(c) $f_X(x) = c/(x^2)$ (d) $f_X(x) = c2^x/x!$

In each case, calculate (where possible) $\mathrm{P}[X > 1]$ and $\mathrm{P}[X \text{ is even}]$

2. n identical fair coins are tossed. Those that show Heads are tossed again, and the number of Heads obtained on the second set of tosses defines a discrete random variable X. Assuming that all tosses are independent, find the range and probability mass function of X.

3. A point is to be selected from an integer lattice restricted to the triangle with corners at $(1,1)$, $(n,1)$ and (n,n) for positive integer n. If all points are equally likely to be selected, find the probability mass functions for the two discrete random variables X and Y corresponding to the $x-$ and $y-$ coordinates of the selected points respectively.

4. A continuous random variable X has pdf given by

$$f_X(x) = c(1-x)x^2 \qquad 0 < x < 1$$

and zero otherwise. Find the value of c, the cdf of X, F_X, and $\mathrm{P}[X > 1/2]$.

5. A function f is defined by

$$f(x) = k/x^{k+1} \qquad x > 1$$

and zero otherwise. For what values of k is f a valid pdf ? Find the cdf of X.

6. A continuous random variable X has pdf given by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and zero otherwise. Sketch f_X, and find the cdf F_X.

7. A continuous random variable X has cdf given by

$$F_X(x) = c(\alpha x^\beta - \beta x^\alpha) \qquad 0 \leq x \leq 1$$

for constants $1 \leq \beta < \alpha$, and zero otherwise.

Find the value of constant c, and evaluate the rth moment of X.

8. A continuous random variable X has cdf given by

$$F_X(x) = \frac{2\beta x}{\beta^2 + x^2} \qquad 0 \leq x \leq \beta$$

for constant $\beta > 0$. Find the pdf of X, and show that the expectation of X is

$$\beta(1 - \log 2)$$