M2S1 : EXERCISE SHEET 4 : SOLUTIONS

1. Using this result for the given joint density, where random variables X, Y take values on the unit square, we have the marginal of U given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}(v, u/v)|v|^{-1}\ dv = \int_u^1 c(\alpha,\beta)\ v^{\alpha-1}(1-v)^{\beta-1}\ \ c(\alpha+\beta,\gamma)\ \left(\frac{u}{v}\right)^{\alpha+\beta-1}\ \left(1-\frac{u}{v}\right)^{\gamma-1} v^{-1}\ dv$$

as we have the density being positive only on the interval $u < v < 1$ for fixed $0 < u < 1$, as $XY < X$. Then

$$\begin{aligned}
f_U(u) &= \int_u^1 c(\alpha,\beta)\ v^{\alpha-1}(1-v)^{\beta-1}\ \ c(\alpha+\beta,\gamma)\ \left(\frac{u}{v}\right)^{\alpha+\beta-1}\ \ \left(1-\frac{u}{v}\right)^{\gamma-1}\frac{1}{v}\ dv \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)\int_u^1 \left(\frac{u}{t}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1} t^{\alpha+\beta-1}(1-t)^{\gamma-1}\ \frac{t}{u}\ \frac{u}{t^2}\ dt \qquad (t = u/v) \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)\int_0^{1-u} \left(\frac{u}{s+u}\right)^{\alpha-1}\left(1-\frac{u}{s+u}\right)^{\beta-1}(s+u)^{\alpha+\beta-2}(1-s-u)^{\gamma-1}\ ds \qquad (s = t-u) \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)\int_0^{1-u} u^{\alpha-1}s^{\beta-1}(1-s-u)^{\gamma-1}\ ds \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)\int_0^1 u^{\alpha-1}r^{\beta-1}(1-u)^{\beta-1}(1-u)^{\gamma-1}(1-r)^{\gamma-1}(1-u)\ dr \qquad (r = s/(1-u)) \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 r^{\beta-1}(1-r)^{\gamma-1}\ dr \\
&= c(\alpha,\beta)c(\alpha+\beta,\gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{1}{c(\beta,\gamma)} = c(\alpha,\beta+\gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0 < u < 1
\end{aligned}$$

2. To compute the covariance need first the marginal expectations of X and Y. The key part of the solution is to realize that the support of the joint density is

$$0 < x < 1, 0 < y < 1, 0 < x+y < 1$$

that is, the "lower left corner" triangle of the unit square, bounded by the three lines $x = 0, y = 0, x + y = 1$. Now, for $0 < x < 1$,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} cxy(1-x-y)\ dy = cx\int_0^{1-x} y(1-x-y)\ dy \\
&= cx(1-x)^3\int_0^1 t(1-t)\ dt \qquad (t = y/(1-x)) \\
&= \frac{c}{6}x(1-x)^3 \qquad 0 < x < 1
\end{aligned}$$

and

$$\int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^1 \frac{c}{6}x(1-x)^3\ dx = 1 \Longrightarrow c = 120$$

and hence

$$f_X(x) = 20x(1-x)^3 \qquad 0 < x < 1 \qquad \Longrightarrow \qquad \mathrm{E}_{f_X}[\ X\ ] = \int_0^1 20x^2(1-x)^3\ dx = \frac{1}{3}$$

and, by symmetry, $f_Y(y) = 20y(1-y)^3$ $(0 < y < 1)$, $\mathrm{E}_{f_Y}[\ Y\ ] = \dfrac{1}{3}$.

by symmetry. Also

$$
\begin{aligned}
\mathrm{E}_{f_{X,Y}}[\,XY\,] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\;dxdy = \int_0^1 \left\{\int_0^{1-y} 120x^2y^2(1-x-y)\;dx\right\} dy \\
&= \int_0^1 120y^2 \left\{\int_0^{1-y} x^2(1-x-y)\;dx\right\} dy \\
&= \int_0^1 120y^2 \left[\frac{x^3}{3}(1-y) - \frac{x^4}{4}\right]_0^{1-y} dy \\
&= \int_0^1 10y^2(1-y)^4\;dy \\
&= 10\left[\frac{y^3}{3} - y^4 + \frac{6y^5}{5} - \frac{4y^6}{6} + \frac{y^7}{7}\right]_0^1 = 10\left(\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right) = \frac{2}{21}
\end{aligned}
$$

and hence

$$
\mathrm{Cov}_{f_{X,Y}}[\,X,Y\,] = \mathrm{E}_{f_{X,Y}}[\,XY\,] - \mathrm{E}_{f_X}[\,X\,]\,.\,\mathrm{E}_{f_Y}[\,Y\,] = \frac{2}{21} - \frac{1}{3}\cdot\frac{1}{3} = -\frac{1}{63}
$$

3. Put $U = X/Y$ and $Z = X$; the inverse transformations are therefore $X = Z$ and $Y = Z/U$, and note that the new variables are constrained by $0 < Z < \min\{U, 1\}$, as $Y < 1$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$
\begin{aligned}
g_1(t_1,t_2) &= t_1/t_2 \qquad & g_1^{-1}(t_1,t_2) &= t_2 \\
g_2(t_1,t_2) &= t_1 \qquad & g_2^{-1}(t_1,t_2) &= t_2/t_1
\end{aligned}
$$

and the Jacobian of the transformation is given by

$$
J(u,z) = \begin{vmatrix} 0 & 1 \\ -z/u^2 & 1/u \end{vmatrix} = \frac{z}{u^2}
$$

and hence

$$
f_{U,Z}(u,z) = f_{X,Y}(z, z/u)\; z/u^2 = z/u^2 \qquad (u,z) \in \mathbb{U}^{(2)} = \{(u,z) : 0 < z < \min\{u,1\}, u > 0\}
$$

and zero otherwise, and so

$$
f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\;dz = \int_0^{\min\{u,1\}} z/u^2\;dz = \frac{(\min\{u,1\})^2}{2u^2} \qquad u > 0.
$$

Put $V = -\log(XY)$ and $Z = -\log X$; the inverse transformations are therefore $X = e^{-Z}$ and $Y = e^{-(v-z)}$, and note that $0 < Z < V$. In terms of the theorem, we have transformation functions defined by

$$
\begin{aligned}
g_1(t_1,t_2) &= -\log(t_1t_2) \qquad & g_1^{-1}(t_1,t_2) &= e^{-t_2} \\
g_2(t_1,t_2) &= -\log t_1 \qquad & g_2^{-1}(t_1,t_2) &= e^{-(t_1-t_2)}
\end{aligned}
$$

and the Jacobian of the transformation is given by

$$
J(v,z) = \begin{vmatrix} 0 & -e^{-z} \\ -e^{-(v-z)} & e^{-(v-z)} \end{vmatrix} = e^{-v}
$$

and hence

$$f_{V,Z}(v,z) = f_{X,Y}(e^{-z}, e^{-(v-z)})\ e^{-v} = e^{-v} \qquad (v,z) \in \mathbb{V}^{(2)} = \{(v,z) : 0 < z < v < \infty\}$$

and zero otherwise, and so

$$f_V(v) = \int_{-\infty}^{\infty} f_{V,Z}(v,z)\ dz = \int_0^v e^{-v}\ dz = ve^{-v} \qquad v > 0$$

and zero otherwise. Note that we can attempt the joint transformation by setting

$$\begin{array}{l} U = X/Y \\ V = -\log(XY) \end{array} \iff \begin{array}{l} X = U^{1/2}e^{-V/2} \\ Y = U^{-1/2}e^{-V/2} \end{array}$$

note that, as X and Y lie in $(0,1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$J(u,v) = \begin{vmatrix} \dfrac{u^{-1/2}e^{-v/2}}{2} & -\dfrac{u^{1/2}e^{-v/2}}{2} \\ -\dfrac{u^{-3/2}e^{-v/2}}{2} & -\dfrac{u^{-1/2}e^{-v/2}}{2} \end{vmatrix} = u^{-1}e^{-v}/2.$$

Hence

$$f_{U,V}(u,v) = u^{-1}e^{-v}/2 \qquad 0 < e^{-v} < \min\{u, 1/u\}\,,\ u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\ dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u}\ dv = \left[-\frac{e^{-v}}{2u}\right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \qquad u > 0 \\ f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\ du = \int_{e^{-v}}^{e^{v}} \frac{e^{-v}}{2u}\ du = \left[\frac{\log u}{2}e^{-v}\right]_{e^{-v}}^{e^{v}} = ve^{-v} \qquad v > 0 \end{aligned}$$

Now let

$$\begin{array}{l} Y = X_1 + X_2 \\ \\ Z = X_1 - X_2 \end{array} \iff \begin{array}{l} X_1 = \dfrac{Y_1 + Y_2}{2} \\ X_2 = \dfrac{Y_1 - Y_2}{2} \end{array}$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square A with corners at $(0,0)$, $(1,1)$, $(0,2)$ and $(1,-1)$ bounded by the lines $z = 1+y$, $z = 1-y$, $z = -1+y$ and $z = -1-y$. Then

$$f_{Y,Z}(y,z) = \frac{1}{2} \qquad (y,z) \in A$$

and zero otherwise (hint: sketch the square A). Hence

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\ dy = \begin{cases} \displaystyle\int_{-z}^{2+z} \frac{1}{2}\ dy = 1 + z & -1 < z \le 0 \\ \displaystyle\int_{z}^{2-z} \frac{1}{2}\ dy = 1 - z & 0 < z < 1 \end{cases}$$

4. The transformations are

$$\begin{array}{l} Y_1 = \dfrac{X_1}{X_1 + X_2 + X_3} \\ Y_2 = \dfrac{X_1}{X_1 + X_2 + X_3} \\ Y_3 = X_1 + X_2 + X_3 \end{array} \iff \begin{array}{l} X_1 = Y_1Y_3 \\ X_2 = Y_2Y_3 \\ X_3 = Y_3(1 - Y_1 - Y_2) \end{array}$$

which gives Jacobian

$$J(y_1,y_2,y_3) = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & (1-y_1-y_2) \end{vmatrix} = y_3^2$$

Hence the joint pdf is given by

$$\begin{aligned} f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) &= f_{X_1,X_2,X_3}(y_1y_3, y_2y_3, y_3(1-y_1-y_2))J(y_1,y_2,y_3) \\ &= c_1 y_1 y_3 \exp\{-y_1y_3\} \;\; c_2 y_2^2 y_3^2 \exp\{-y_2y_3\} \;\; c_3 y_3^3 (1-y_1-y_2)^3 \exp\{-y_3(1-y_1-y_2)\} \;\; y_3^2 \\ &= c_1c_2c_3 y_1 y_2^2 (1-y_1-y_2)^3 \;\; y_3^8 \exp\{-y_3\} = f_{Y_1,Y_2}(y_1,y_2) f_{Y_3}(y_3) \end{aligned}$$

where

$$f_{Y_1,Y_2}(y_1,y_2) \propto y_1 y_2^2 (1-y_1-y_2)^3. \quad \text{and} \quad f_{Y_3}(y_3) \propto y_3^8 \exp\{-y_3\}$$

Hence $Y_3 \sim Gamma(9,1)$; the transformations give the constraints $0 < Y_1, Y_2 < 1$ and $0 < Y_1 + Y_2 < 1$, and $Y_3 > 0$. Now

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_0^{1-y_1} c y_1 y_2^2 (1-y_1-y_2)^3\,dy_2 = c y_1 (1-y_1)^6 \int_0^1 t^2(1-t)^3\,dt \quad (t = y_2/(1-y_1))$$

and hence

$$f_{Y_1}(y_1) \propto y_1(1-y_1)^6 \Longrightarrow Y_1 \sim Beta(2,7), \;\; f_{Y_1}(y_1) = 336 y_1 (1-y_1)^6 \qquad 0 < y_1 < 1$$

and hence

$$\mathrm{E}_{f_{Y_1}}[\,Y_1\,] = \frac{2}{2+7} = \frac{2}{9}$$

as the expectation of a $Beta(\alpha,\beta)$ distribution is $\alpha/(\alpha+\beta)$ from notes.

5. Put $U = X/Y$ and $V = Y$; the inverse transformations are therefore $X = UV$ and $Y = V$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$\begin{aligned} g_1(t_1,t_2) &= t_1/t_2 \qquad & g_1^{-1}(t_1,t_2) &= t_1t_2 \\ g_2(t_1,t_2) &= t_2 \qquad & g_2^{-1}(t_1,t_2) &= t_2 \end{aligned}$$

and the Jacobian of the transformation is given by

$$|J(u,v)| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

and hence

$$f_{U,V}(u,v) = f_{X,Y}(uv,v)\;\;|v| = \left(\frac{1}{2\pi}\right)\exp\left\{-\frac{1}{2}(u^2v^2+v^2\right\}|v| \qquad (u,v) \in \mathbb{R}^2$$

and zero otherwise, and so, for any real u,

$$\begin{aligned} f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)\exp\left\{-\tfrac{1}{2}(u^2v^2+v^2\right\}|v|\;\,dv \\ &= \left(\frac{1}{\pi}\right)\int_0^{\infty} v\exp\left\{-\frac{v^2}{2}(1+u^2)\right\}\,dv \qquad & \text{integrand is even function} \\ &= \left(\frac{1}{\pi}\right)\left[-\frac{1}{(1+u^2)}\exp\left\{-\frac{v^2}{2}(1+u^2)\right\}\right]_0^{\infty} \qquad & \text{by direct integration} \\ &= \frac{1}{\pi(1+u^2)} \end{aligned}$$

Now put $T = X/\sqrt{S/\nu}$ and $R = S$; the inverse transformations are therefore $X = T\sqrt{R/\nu}$ and $S = R$. In terms of the multivariate transformation theorem, we have transformation functions from $(X, S) \to (T, R)$ defined by

$$g_1(t_1, t_2) = t_1/\sqrt{t_2/\nu} \qquad g_1^{-1}(t_1, t_2) = t_1\sqrt{t_2/\nu}$$

$$g_2(t_1, t_2) = t_2 \qquad g_2^{-1}(t_1, t_2) = t_2$$

and the Jacobian of the transformation is given by

$$|J(t,r)| = \left| \begin{array}{cc} \sqrt{\dfrac{r}{\nu}} & \dfrac{t}{2\sqrt{r\nu}} \\ 0 & 1 \end{array} \right| = \left|\sqrt{\frac{r}{\nu}}\right| = \sqrt{\frac{r}{\nu}}$$

and hence

$$f_{T,R}(t,r) = f_{X,S}\left(t\sqrt{\frac{r}{\nu}}, r\right)\sqrt{\frac{r}{\nu}} \quad = \quad f_X\left(t\sqrt{\frac{r}{\nu}}\right) f_S(r)\sqrt{\frac{r}{\nu}} \qquad t \in \mathbb{R}, s \in \mathbb{R}^+$$

and zero otherwise, and so, for any real t,

$$\begin{aligned}
f_T(t) &= \int_{-\infty}^{-\infty} f_{T,R}(t,r)\, dr \\
&= \int_0^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{rt^2}{2\nu}\right\} c(\nu) r^{\nu/2-1} e^{-r/2}\sqrt{\frac{r}{\nu}}\, dr \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \int_0^{\infty} r^{(\nu+1)/2-1} \exp\left\{-\frac{r}{2}\left(1+\frac{t^2}{\nu}\right)\right\} dr \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \int_0^{\infty} z^{(\nu+1)/2-1} \exp\left\{-\frac{z}{2}\right\} dz \quad \text{setting } z = r\left(1+\frac{t^2}{\nu}\right) \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \frac{1}{c(\nu+1)} \qquad \text{integrand is a pdf}
\end{aligned}$$

We also see/deduce that f_S is a $Gamma(\nu/2, 1/2)$ or $Chiquared(\nu)$ density, and that the normalizing constant $c(\nu)$ is given by

$$c(\nu) = \frac{\left(\dfrac{1}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} \qquad \Longrightarrow \qquad f_T(t) = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)}\left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$$

which is the $Student(\nu)$ density.

6. We have

$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \qquad x \in \mathbb{R} \qquad\qquad f_Y(y) = c(\nu) y^{\nu/2-1} e^{-\nu y/2} \qquad y \in \mathbb{R}^+$$

where ν is a positive integer, so that $X|Y = y \sim N(0, y^{-1})$ and $Y \sim Gamma(\nu/2, \nu/2)$, and the normalizing constant $c(\nu)$ is given by

$$c(\nu) = \frac{\left(\dfrac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)}$$

Now, by the chain rule

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) \qquad x \in \mathbb{R}, y \in \mathbb{R}^+$$

and zero otherwise, and so, for any real x,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dy \\
&= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\nu/2-1} e^{-\nu y/2}\; dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} y^{(\nu+1)/2-1} \exp\left\{-\frac{y}{2}\left(\nu + x^2\right)\right\}\; dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{1}{2}\left(\nu + x^2\right)\right)^{(\nu+1)/2}} \qquad \text{integrand} \propto \text{a Gamma pdf}
\end{aligned}$$

Therefore f_X is given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1 + x^2/\nu)^{(\nu+1)/2}}$$

which is again the $Student(\nu)$ density.

Exercises 5 and 6 give the two alternative ways of specifying the Student-t distribution, either as a function of independent Normal and Gamma/Chi-squared variables, or as the marginal obtained by "*scale-mixing*" a Normal distribution by a Gamma distribution (that is, rather than having a fixed variance $\sigma^2 = 1/Y$,.we regard Y as a *random variable* having a Gamma distribution, so that (X, Y) have a joint distribution

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

from which we calculate $f_X(x)$ by integration.

7. Can calculate the mgf for this mass function (the Poisson distribution) as

$$M_X(t) = \exp\left\{\lambda(e^t - 1)\right\}.$$

Now, if $Z_1 = (X - \lambda)/\sqrt{\lambda}$, we use the mgf result for linear functions, that is if $Y = aX + b$, $M_X(t) = e^{bt} M_X(at)$. Here, $a = 1/\sqrt{\lambda}$ and $b = -\sqrt{\lambda}$, so

$$\begin{aligned}
M_{Z_1}(t) &= e^{-\sqrt{\lambda}t} \exp\left\{\lambda(e^{t/\sqrt{\lambda}} - 1)\right\} \quad = \quad \exp\left\{-\sqrt{\lambda}t + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\,{}^{3/2}\!\sqrt{\lambda}} + \ldots\right]\right\} \\
&= \exp\left\{\frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \ldots\right\} \to \exp\left\{\frac{t^2}{2}\right\} \qquad \text{as } \lambda \to \infty
\end{aligned}$$

so therefore

$$Z_1 \to Z \sim Normal(0,1) \qquad \text{as } \lambda \to \infty$$

that is, the distribution Z_1tends to a standard Normal distribution as $\lambda \to \infty$. Re-arranging the definition of Z_1, and using the transformation result for Normal random variables

$$X \sim Normal(0,1), Y = aX + b \Longrightarrow Y \sim Normal(b, a^2)$$

we have that

$$X = \sqrt{\lambda} Z_1 + \lambda \sim Normal(\lambda, \lambda)$$

Thus we have that

$$Poisson(\lambda) \approx Normal(\lambda, \lambda) \qquad \text{if } \lambda \text{ is large}$$

For second part, can calculate the mgf for this pdf function (the Gamma distribution with parameters λ and 1) as

$$M_X(t) = \left(\frac{1}{1-t}\right)^{\lambda}.$$

Now, if $Z_2 = (X-\lambda)/\sqrt{\lambda}$, we use the mgf result for linear functions, that is if $Y = aX+b, M_X(t) = e^{bt}M_X(at)$. Here, $a = 1/\sqrt{\lambda}$ and $b = -\sqrt{\lambda}$, so

$$\begin{aligned} M_{Z_2}(t) &= e^{-\sqrt{\lambda}t}\left(\frac{1}{1-t/\sqrt{\lambda}}\right)^{\lambda} = \exp\left\{-\sqrt{\lambda}t - \lambda\log\left(1-\frac{t}{\sqrt{\lambda}}\right)\right\} \\ &= \exp\left\{-\sqrt{\lambda}t + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\ {}^{3/2}\!\sqrt{\lambda}} + ...\right]\right\} \\ &= \exp\left\{\frac{t^2}{2} + \frac{t^3}{3\sqrt{\lambda}} + ...\right\} \to \exp\left\{\frac{t^2}{2}\right\} \qquad \text{as } \lambda \to \infty \end{aligned}$$

so therefore

$$Z_2 \to Z \sim Normal(0,1) \qquad \text{as } \lambda \to \infty$$

that is, the distribution Z_2tends to a standard Normal distribution as $\lambda \to \infty$. Re-arranging the definition of Z_2, and using the transformation result for Normal random variables as above, we have that

$$X = \sqrt{\lambda}Z_2 + \lambda \sim Normal(\lambda, \lambda)$$

Thus we have that

$$Gamma(\lambda, 1) \approx Normal(\lambda, \lambda) \qquad \text{if } \lambda \text{ is large}$$

8. If $Y_2 = \max\{X_1, X_2\}$ then P[$Y_2 > c$] $= 1 - p^2$, as

$$\text{P}[\ Y_2 \le c\ ] = \text{P}[\ \max\{X_1, X_2\} \le c\ ] = \text{P}[\ (X_1 \le c) \cap (X_2 \le c)\ ] = \text{P}[\ X_1 \le c\ ]\text{P}[\ X_2 \le c\ ] = p^2$$

9. If $Y_1 = \min\{X_1, ..., X_k\}$, then

$$\begin{aligned} F_{Y_1}(y_1) &= \text{P}[\ Y_1 \le y_1\ ] = \text{P}[\ \min\{X_1,...,X_k\} \le y_1\ ] \\ &= 1 - \text{P}[\ \min\{X_1,...,X_k\} > y_1\ ] = 1 - \text{P}[\ (X_1 > y_1) \cap ... \cap (X_k > y_1)\ ] \\ &= 1 - \text{P}[\ (X_1 > y_1)\ ]...\text{P}[\ (X_k > y_1)\ ] = 1 - (1 - F_X(y_1))...(1-F_X(y_1)) = 1 - \{(1-F_X(y_1))\}^k \end{aligned}$$

Now here, $f_X(x) = \lambda e^{-\lambda x}$, so $F_X(x) = 1 - e^{-\lambda x}$, when $x > 0$, and hence

$$F_{Y_1}(y_1) = 1 - \{(1-F_X(y_1))\}^k = 1 - \left\{1-(1-e^{-\lambda y_1})\right\}^k = 1 - e^{-\lambda y_1} \quad \Longrightarrow \quad f_{Y_1}(y_1) = k\lambda e^{-k\lambda y_1} \qquad y_1 > 0.$$

.In part two, the cdf for X is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_1^x \frac{1}{t^2}dt = 1 - \frac{1}{x} = \frac{x-1}{x} \qquad 1 \le x \le \infty$$

Using Result 1 of the Order Statistics section: the joint distribution of the order statistics $Y_1, ..., Y_k$ derived from $X_1, ..., X_k$ is

$$f_{Y_1,...,Y_k}(y_1,...,y_k) = k! f_{X_1,...,X_k}(x_1,...,x_k) = k!\prod_{i=1}^{k}\frac{1}{y_i^2}$$

and by Result 2, or by direct calculation as above, the marginal pdfs of minimum Y_1 and maximum Y_k are given by

$$f_{Y_1}(y_1) = k\left\{\frac{1}{y_1}\right\}^{k-1}\frac{1}{y_1^2} = \frac{k}{y_1^{k+1}} \qquad f_{Y_k}(y_k) = k\left\{\frac{y_k - 1}{y_k}\right\}^{k-1}\frac{1}{y_k^2} = \frac{k(y_k-1)^{k-1}}{y_k^{k+1}}$$

with support $[1,\infty)$ in each case.

10. We have $F_X(x) = x$, and $f_X(x) = 1$ for $0 < x < 1$. Either by direct calculation, or by using the Order Statistics Result 2, we have

$$F_{Y_k}(y_k) = \{F_X(y_k)\}^k = y_k^k \qquad f_{Y_k}(y_k) = \frac{k!}{0!\,(k-1)!}\{F_X(y_k)\}^{k-1}\{1-F_X(y_k)\}^0 f_X(y_k) = ky_k^{k-1}$$

so that

$$\mathrm{P}[\,Y_k \geq 0.99\,] = 1 - \mathrm{P}[\,Y_k < 0.99\,] = 1 - F_{Y_k}(0.99) = 1 - (0.99)^k$$

and the smallest k such that $1-(0.99)^k \geq 0.95$ is $k = 295$.

11. For this pdf, we have cdf $F_X(x) = x^5,\ 0 < x < 1$. From elementary calculations, we have that

$$F_{Y_1}(y_1) = 1 - \{1 - F_X(y_1)\}^k = 1 - (1-y_1^5)^k \qquad 0 < y_1 < 1.$$

Therfore, $\mathrm{P}[\,Y_1 \leq 0.75\,] = 1 - (1-(0.75)^5)^k$, which takes the values 0.556, 0.662, 0.742 for $k = 3, 4, 5$. Also

$$f_{Y_1}(y_1) = k\{1 - F_X(y_1)\}^{k-1} f_X(y_1) = 5ky_1^4\left\{1 - y_1^5\right\}^{k-1} \qquad 0 \leq y_1 \leq 1$$

and

$$F_{Y_1}(y_1) = 1 - \left\{1 - y_1^5\right\}^k = 1 - (1-y_1^5)^k \quad \longrightarrow \quad \begin{cases} 0 & y_1 = 0 \\ & \\ 1 & y_1 > 1 \end{cases}$$

as $k \longrightarrow \infty$, so that the limiting distribution is a step function, with the step at zero. That is, the limiting distribution has mass one at $Y_1 = 0$.

12. We have that

$$f_X(x) = \frac{(\alpha+1)x^\alpha}{\theta^{\alpha+1}} \qquad F_X(x) = \left(\frac{x}{\theta}\right)^{\alpha+1} \qquad 0 \leq x \leq \theta$$

and using the results for maxima and minima, we have that

$$\begin{aligned} F_{Y_1}(y) &= 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(\frac{y}{\theta}\right)^{\alpha+1}\right\}^n \qquad 0 \leq x \leq \theta \\ &= \left\{\begin{array}{ll} 0 & y < 0 \\ 1 & y \geq 0 \end{array}\right. \qquad \text{as } n \to \infty \text{ with } y \textbf{ fixed}. \end{aligned}$$

(as the second term is 1 if $y = 0$, but is zero otherwise) that is, the limiting distribution (cdf) as $n \to \infty$ is a step function, with a single step at $y = 0$. Hence the limiting distribution is *degenerate* at $y = 0$, that is, we have Y_1 converging in distribution to a discrete random variable Y with $\mathrm{P}[Y = 0] = 1$.

Similarly, for the maximum order statistic,Y_n we have

$$\begin{aligned} F_{Yn}(y) &= \{F_X(y)\}^n = \left\{\left(\frac{y}{\theta}\right)^{\alpha+1}\right\}^n = \left(\frac{y}{\theta}\right)^{n(\alpha+1)} \qquad 0 \leq x \leq \theta \\ &= \left\{\begin{array}{ll} 0 & y < \theta \\ 1 & y \geq \theta \end{array}\right. \qquad \text{as } n \to \infty \text{ with } y \textbf{ fixed} \end{aligned}$$

(as y/θ is less than 1 if $y < \theta$) that is, the limiting distribution (cdf) as $n \to \infty$ is a step function, with a single step at $y = \theta$. Hence the limiting distribution is *degenerate* at $y = \theta$, that is, we have Y_n converging in distribution to a discrete random variable Y with $\mathrm{P}[Y = \theta] = 1$.