M2S1 : EXERCISE SHEET 3 : SOLUTIONS

1 To calculate the mgf

$$
\begin{aligned}
M_X(t) &= \mathrm{E}_{f_X}[\ e^{tX}\ ] = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\ dx = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-t)^2}{2}\right\}\ dx \\
&= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}\ dy = e^{t^2/2}
\end{aligned}
$$

completing the square in x, and then setting $y = x - t$, as the integrand is a pdf.
Now, using the transformation theorem for univariate, 1-1 transformations we have $Y = \sigma X + \mu \iff X = (Y - \mu)/\sigma$, so

$$
f_Y(y) = f_X((y-\mu)/\sigma)\ .\ \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \qquad y \in \mathbb{R}
$$

To calculate the mgf of Y, use the mgf result given in lectures

$$
M_Y(t) = e^{\mu t} M_X(\sigma t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}
$$

If $T = X^2$, then from first principles

$$
\begin{aligned}
F_T(t) &= \mathrm{P}[\ T \leq t\ ] = \mathrm{P}[\ X^2 \leq t\ ] = \mathrm{P}[\ -\sqrt{t} \leq X \leq \sqrt{t}\ ] \\
\Longrightarrow f_T(t) &= \frac{1}{2\sqrt{t}} \left[f_X(\sqrt{t}) + f_X(-\sqrt{t})\right] = \frac{1}{\sqrt{2\pi}} t^{-1/2} \exp\left\{-\frac{t}{2}\right\} \qquad t > 0
\end{aligned}
$$

and hence

$$
\begin{aligned}
M_T(t) &= \mathrm{E}_{f_T}[\ e^{tT}\ ] = \int_{-\infty}^{\infty} e^{tx} f_T(x)\ dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{x}{2}\right\}\ dx = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{(1-2t)x}{2}\right\}\ dx \\
&= \left(\frac{1}{1-2t}\right)^{1/2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\}\ dy = \left(\frac{1}{1-2t}\right)^{1/2}
\end{aligned}
$$

where $y = (1-2t)x$, as the integrand is a pdf.

2. By definition of mgfs for discrete variables, we can deduce immediately that, as

$$
M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x)
$$

$\mathrm{P}[\ X = x\ ]$ is just the coefficient of e^{tx} in the expression for M_X, and hence $\mathrm{P}[\ X = 1\ ] = 1/8$, $\mathrm{P}[\ X = 2\ ] = 1/4$ and $\mathrm{P}[\ X = 3\ ] = 5/8$. Also, we have $\mathrm{E}_{f_X}[\ X^r\ ] = M_X^{(r)}(0)$, so that

$$
\mathrm{E}_{f_X}[\ X\ ] = M_X^{(1)}(0) = \frac{1}{8} + 2\frac{1}{4} + 3\frac{5}{8} = \frac{5}{2} \qquad \mathrm{E}_{f_X}[\ X^2\ ] = M_X^{(2)}(0) = \frac{1}{8} + 4\frac{1}{4} + 9\frac{5}{8} = \frac{27}{4}
$$

$$
\Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] = \mathrm{E}_{f_X}[\ X^2\ ] - \{\mathrm{E}_{f_X}[\ X\ ]\}^2 = \frac{1}{2}
$$

3. If $X \sim Bin(n, \theta)$,

$$
M_X(t) = (1 - \theta + \theta e^t)^n = (1 + (e^t - 1)\theta)^n = \left(1 + \theta\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)\right)^n
$$

and from the mgf definition $\mathrm{E}_{f_X}[\ X^r\ ]$ is $r!$ times the coefficient of t^r in this expansion. Difficult to identify this general term, but can easily identify the coefficient of t as $n\theta = \mathrm{E}_{f_X}[\ X\ ]$, and the coefficient of t^2 as $n\theta + n(n-1)\theta^2 = \mathrm{E}_{f_X}[\ X^2\ ]$ etc.

4. For this pdf,

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\ dx = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\ dx = e^{-2}\int_{-2}^{\infty} e^{-(1-t)x}\ dx \\ &= \frac{e^{-2}}{1-t}\int_{-2(1-t)}^{\infty} e^{-y}\ dy \ = \ \frac{e^{-2}}{1-t}\ \left[-e^{-y}\right]_{-2(1-t)}^{\infty} \ = \ \frac{e^{-2t}}{1-t} \qquad t<1 \end{aligned}$$

Now

$$\begin{aligned} M_X^{(1)}(t) &= \frac{e^{-2t}}{(1-t)^2}(2t-1) \Longrightarrow M_X^{(1)}(0) = -1 = \mathrm{E}_{f_X}[\ X\ ] \\ M_X^{(2)}(t) &= \frac{e^{-2t}}{(1-t)^3}\left[1+(2t-1)^2\right] \Longrightarrow M_X^{(2)}(0) = 2 = \mathrm{E}_{f_X}[\ X^2\ ] \Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] = 1 \end{aligned}$$

5. We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathrm{E}_{f_X}[\ X\ ]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = \mathrm{E}_{f_X}[\ X^2\ ] - \left\{\mathrm{E}_{f_X}[\ X\ ]\right\}^2 = \mathrm{Var}_{f_X}[\ X\ ]$$

6. Easy to see that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, with $\mathbb{X}^{(2)} = \mathbb{X}\times\mathbb{Y}$, so X and Y are independent, where

$$f_X(x) = \sqrt{c}\ \frac{2^x}{x!} \qquad f_Y(y) = \sqrt{c}\ \frac{2^y}{y!} \qquad \text{and} \qquad \sum_{x=0}^{\infty} f_X(x) = 1 \Longrightarrow \sqrt{c} = e^{-2}$$

(marginal mass functions must have identical forms as joint mass function is symmetric in x and y) as the summation is identical to the power series expansion of e^z at $z=2$ if $\sqrt{c} = e^{-2}$.

7. $F_{X,Y}$ is continuous and non decreasing in x and y, and

$$\lim_{x\longrightarrow-\infty} F_{X,Y}(x,y) = \lim_{y\longrightarrow-\infty} F_{X,Y}(x,y) = 0 \qquad \lim_{x,y\longrightarrow\infty} F_{X,Y}(x,y) = 1$$

so $F_{X,Y}$ is a valid cdf, and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\left\{F_{X,Y}(t_1,t_2)\right\}_{t_1=x,t_2=y} = \frac{e^{-x}}{\pi(1+y^2)} = f_X(x)f_Y(y)$$

so as $\mathbb{X}^{(2)} = \mathbb{R}^+\times\mathbb{R}$, X and Y are independent.

8. The form of the joint range $\mathbb{X}^{(2)}$ is the key point; we have $\mathbb{X}^{(2)} = \{\ (x, y)\ :\ x > 0, 0 < y < \exp\{-\beta x^\alpha\}\}$, and hence

$$
\begin{aligned}
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy &= \int_0^{e^{-\beta x^\alpha}} cx^{\alpha-1}\ dy = cx^{\alpha-1}\exp\{-\beta x^\alpha\} \qquad x > 0 \\
\Longrightarrow F_X(x) &= \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x ct^{\alpha-1}\exp\{-\beta t^\alpha\} = \frac{c}{\alpha\beta}\left(1 - \exp\{-\beta x^\alpha\}\right)
\end{aligned}
$$

so that $c = \alpha\beta$. Similarly, letting $g(y) = \{-\log y/\beta\}^{1/\alpha}$, we have $0 < x < g(y)$ as $0 < y < \exp\{-\beta x^\alpha\}$, and

$$
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^{g(y)} cx^{\alpha-1}\ dx = \frac{c}{\alpha}\{g(y)\}^\alpha = -\log y \qquad 0 < y < 1
$$

9. (a) We have

$$
\frac{\partial^2}{\partial t_1 \partial t_2}\{F_1(t_1,t_2)\}_{t_1=x,t_2=y} = -e^{-x-y} < 0
$$

on the specified range of X and Y, so $F_{X,Y}$ is not a valid cdf, as this partial derivative must be non-negative.

(b) We have

$$
\frac{\partial^2}{\partial t_1 \partial t_2}\{F_2(t_1,t_2)\}_{t_1=x,t_2=y} = \begin{cases} e^{-y} & 0 \leq x \leq y \\ e^{-x} & 0 \leq y \leq x \end{cases}
$$

which is non-negative everywhere. Note that, $F_2(x, 0) = F_2(0, y) = 0$. However, consider the behaviour of F_2 as x and y become large; first, consider the cdf F defined by

$$
F(x,y) = 1 - e^{-y} - ye^{-x} \qquad 0 \leq y \leq x < \infty,
$$

that is, identical to F_2 on only half the original domain. It is easy to check that F is a cdf, in particular, that

$$
\lim_{x,y\longrightarrow\infty} F(x,y) = 1
$$

Hence, by symmetry we must have that

$$
\lim_{x,y\longrightarrow\infty} F_2(x,y) = 2,
$$

so F_2 is not a valid cdf.

Changing the question slightly gives a different solution; if cdf F_2 is defined as

$$
F_2(x,y) = \begin{cases} 1 - e^{-x} - xe^{-y} & 0 \leq x \leq y \\ 1 - e^{-y} - ye^{-y} & 0 \leq y \leq x \end{cases}
$$

then we have

$$
f_2(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\{F_2(t_1,t_2)\}_{t_1=x,t_2=y} = e^{-y} \qquad 0 \leq x \leq y < \infty
$$

and zero otherwise, which is also non-negative everywhere. Again, $F_2(x, 0) = F_2(0, y) = 0$, and here

$$
\lim_{x,y\longrightarrow\infty} F_2(x,y) = 1
$$

so F_2 **is** a valid cdf.

Note also that, in the amended question the marginal cdfs for X and Y are given by

$$\begin{aligned} F_X(x) &= \lim_{y\longrightarrow\infty} F_2(x,y) = 1-e^{-x} \qquad x \geq 0 \\ F_Y(y) &= \lim_{x\longrightarrow\infty} F_2(x,y) = 1-(y+1)e^{-y} \qquad y \geq 0 \end{aligned}$$

10. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$. The marginals are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{1/x}^{x} \frac{1}{2x^2 y}\, dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\, dy = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\, dy = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y\log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

Marginal expectation of Y;

$$\mathrm{E}_{f_Y}[\,Y\,] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 \frac{y}{2}\, dy + \int_1^{\infty} \frac{1}{2y}\, dy = \infty$$

as the second integral is divergent.

11. To compute c,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\, dxdydz = \int_0^1\int_0^z\int_0^y c\, dxdydz = c\int_0^1\int_0^z y\, dydz = c\int_0^1 z^2/2\, dz = c/6$$

so $c = 6$.

$$
\begin{aligned}
f_{X,Z}(x,z) &= \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy = \int_x^z 6\ dy &&= 6(z-x) \quad 0<x<z<1 \\
f_{Y,Z}(y,z) &= \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dx = \int_0^y 6\ dx &&= 6y \quad 0<y<z<1 \\
f_{Y|X,Z}(y|x,z) &= \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Z}(x,z)} &&= \frac{1}{z-x} \quad x<y<z \\
f_{X|Y,Z}(x|y,z) &= \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} &&= \frac{1}{y} \quad 0<x<y \\
f_{X,Y|Z}(x,y|z) &= \frac{f_{X,Y,Z}(x,y,z)}{f_{Z}(z)} &&= \frac{2}{z^2} \quad x<y<z
\end{aligned}
$$

as

$$
f_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy\ dx = \int_0^z\int_0^y 6\ dxdy = 3z^2
$$

12.

We have that

$$
M_{X,Y}(t,u) = \int\int e^{tx+uy} f_{X,Y}(x,y)\ dxdy
$$

so

$$
M_{X,Y}(t,0) = \int e^{tx}\left\{\int f_{Y|X}(y|x)\ dy\right\} f_X(x)\ dx
$$

by the chain rule for random variables. Hence, as the internal integral must be equal to 1 (integrand is a conditional pdf), we have

$$
M_{X,Y}(t,0) = \int e^{tx} f_X(x)\ dx = M_X(t)
$$

and similarly for $M_{X,Y}(0,u) = M_Y(u)$. Moments follow by appropriate partial differentiation inside the integral, and evaluation at $t = u = 0$.

Hence the terms in

$$
\mathrm{Cov}_{f_{X,Y}}[X,Y] = \mathrm{E}_{f_{X,Y}}[XY] - \mathrm{E}_{f_X}[X]\,\mathrm{E}_{f_Y}[Y]
$$

are given by

$$
\mathrm{E}_{f_{X,Y}}[XY] = \frac{\partial^2}{\partial t\partial u}\{M_{X,Y}(t,u)\}_{t=0,u=0} = M_{X,Y}^{(1,1)}(0,0)
$$

and

$$
\mathrm{E}_{f_X}[X] = \frac{\partial}{\partial t}\{M_{X,Y}(t,0)\}_{t=0} = M_{X,Y}^{(1,0)}(0,0) \qquad \mathrm{E}_{f_Y}[Y] = \frac{\partial}{\partial u}\{M_{X,Y}(0,u)\}_{u=0} = M_{X,Y}^{(0,1)}(0,0)
$$

where

$$
M_{X,Y}^{(j,k)}(s_1,s_2) = \frac{\partial^{j+k}}{\partial t^j\partial u^k}\{M_{X,Y}(t,u)\}_{t=s_1,u=s_2}.
$$