M2S1 : ASSESSED COURSEWORK 2 : SOLUTIONS

1.First note that $f_X(x)$ is symmetric about $x = \mu$, and also that as $f_X(x)$ is a pdf we have that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(x-\mu)^2\right\}\,dx = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,dt = 1 \qquad (1)$$

using the substitution $t = \lambda^{1/2}(x-\mu)$ in the integral Note also that therefore

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{t^2}{2}\right\}\,dt = \sqrt{2\pi}.$$

(a) The expectation of X is

$$\begin{aligned}
\mathrm{E}_{f_X}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \quad = \quad \int_{-\infty}^{\infty} x \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(x-\mu)^2\right\}\,dx \\
&= \int_{-\infty}^{\infty} \left(\mu + t\lambda^{-1/2}\right) \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \lambda^{-1/2}\,dt \qquad t = \lambda^{1/2}(x-\mu) \\
&= \mu \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,dt + \lambda^{-1/2} \int_{-\infty}^{\infty} t \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\,dt \\
&= \mu
\end{aligned}$$

as the first integral is 1 from (1), and the second integral is zero, as the integrand is an ODD function about zero. Hence

$$\mathrm{E}_{f_X}[X] = \mu$$

and note that it is generally true that if a pdf is symmetric about a particular value, then that value is the expectation.

[3 MARKS]

(b) The expectation of $g(X) = e^X$ is

$$\begin{aligned}
\mathrm{E}_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} e^x \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(x-\mu)^2\right\}\,dx \\
&= \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1/2}\right\} \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \lambda^{-1/2}\,dt \quad t = \lambda^{1/2}(x-\mu) \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1/2} - \frac{t^2}{2}\right\}\,dt \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t^2 - 2t\lambda^{-1/2} - 2\mu\right)\right\}\,dt
\end{aligned}$$

Completing the square in the exponent, we have

$$\left(t^2 - 2t\lambda^{-1/2} + 2\mu\right) = \left(t - \lambda^{-1/2}\right)^2 - \left(2\mu + \lambda^{-1}\right)$$

and hence

$$\begin{aligned}
\mathrm{E}_{f_X}\left[g(X)\right] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t-\lambda^{-1/2}\right)^2 + \left(\mu + \frac{1}{2\lambda}\right)\right\} \, dt \\
&= \exp\left\{\mu + \frac{1}{2\lambda}\right\} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(t-\lambda^{-1/2}\right)^2\right\} \, dt \\
&= \exp\left\{\mu + \frac{1}{2\lambda}\right\}
\end{aligned}$$

as the integral is equal to 1, using (1), which holds for all choices of μ and λ.

[*4 MARKS*]

(c) If $Y = e^X$, so $\mathbb{Y} = \mathbb{R}^+$, and from first principles we have

$$F_Y(y) = \mathrm{P}\left[Y \le y\right] = \mathrm{P}\left[e^X \le y\right] = \mathrm{P}\left[X \le \log y\right] = F_X(\log y)$$

and hence, by differentiation,

$$f_Y(y) = f_X(\log y)\,\frac{1}{y} \qquad\qquad y > 0$$

(note: the function $g(t) = e^t$ is a monotone increasing function, with $g^{-1}(t) = \log t$, so that we can use the general result directly, that is

$$f_Y(y) = f_X(g^{-1}(y)) \; |\, J(y)\,| \qquad \text{where} \qquad J(y) = \frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y} = \frac{d}{dt}\left\{\log t\right\}_{t=y} = \frac{1}{y}$$

and so on). Hence

$$f_Y(y) = \frac{1}{y}\left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(\log y - \mu)^2\right\} \qquad\qquad y > 0.$$

[*2 MARKS*]

For the expectation, we have from first principles

$$\begin{aligned}
\mathrm{E}_{f_Y}\left[Y\right] &= \int_0^{\infty} y f_Y(y) \, dy = \int_{-\infty}^{\infty} y\,\frac{1}{y}\left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(\log y - \mu)^2\right\} \, dy \\
&= \int_{-\infty}^{\infty} \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(t - \mu)^2\right\} e^t \, dt \;\; = \;\; \exp\left\{\mu + \frac{1}{2\lambda}\right\}
\end{aligned}$$

where $t = \log y$, as the integral is precisely the one carried out above. This illustrates the transformation/expectation result that, if $Y = g(X)$, then

$$\mathrm{E}_{f_Y}\left[Y\right] = \mathrm{E}_{f_X}\left[g(X)\right]$$

Mark also given for spotting this result without giving working.

[*1 MARK*]

2.(i) If $\mathbb{X}^{(2)} = (0,1) \times (0,1)$ is the (joint) range of vector random variable (X, Y). We have

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

so that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \qquad \text{and} \qquad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the ranges of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1-y) \tag{2}$$

for some constants satisfying $c_1 c_2 = c$. Hence, the two conditions for independence are satisfied in (2), and X and Y are independent.

[*2 MARKS*]

(ii) We must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1 \therefore c^{-1} = \int_0^1\int_0^1 x(1-y)\ dxdy = 1$$

and as

$$\int_0^1\int_0^1 x(1-y)\ dxdy = \left\{\int_0^1 x\ dx\right\}\left\{\int_0^1 (1-y)\ dy\right\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

[*2 MARKS*]

(iii) We have $A = \{(x,y)\ :\ 0 < x < y < 1\}$, and hence

$$\begin{aligned}
\mathrm{P}[\,X < Y\,] &= \int_A\int f_{X,Y}(x,y)\ dxdy \\
&= \int_0^1 \left\{\int_0^y 4x(1-y)\ dx\right\} dy \qquad \text{(joint density is only non-zero when } x < y \\
&\qquad\qquad\qquad\qquad\qquad\quad \text{- hence for fixed } y\text{, we integrate } dx \text{ on the range } (0,y)) \\
&= \int_0^1 \left\{\int_0^y x\ dx\right\} 4(1-y)\ dy \\
&= \int_0^1 2y^2(1-y)\ dy \\
&= \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6}
\end{aligned}$$

[*3 MARKS*]

(b) Now suppose the joint pdf of X and Y is given by

$$f_{X.Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise, the marginal pdf f_X is given by

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy \\
&= \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} \\
&= 12x(1-x)^2 \qquad\qquad 0 < x < 1
\end{aligned}
$$

as the integrand is only non-zero when $0 < x + y < 1 \Longrightarrow 0 < y < 1 - x$ for fixed x

[3 MARKS]

MAPLE IMPLEMENTATION.