M2S1 : ASSESSED COURSEWORK 1 : SOLUTIONS

1.Given that there are n defective links, with associated independent failure time random variables $Y_1, ..., Y_n$, we have

$$\begin{aligned} r_{C\,|\,N}(y\,|\,n) &= P(\text{Chain breaks later than time } y\,|\,N=n) \\ &= P(Y_1 > y, Y_2 > y, ..., Y_n > y\,|\,N=n) && \text{(chain breaks when first link breaks)} \\ &= \prod_{i=1}^{n} P[Y_i > y] && \text{(by independence)} \\ &= \prod_{i=1}^{n} r_L(y) = \prod_{i=1}^{n} (1 - F_L(y)) && \text{(as links are identical)} \\ &= \exp\{-n\lambda y^{\alpha}\} \end{aligned}$$

Hence required probability is

$$1 - r_{C\,|\,N}(y\,|\,n) = 1 - \exp\{-n\lambda y^{\alpha}\}.$$

[5 MARKS]

Hence, using the theorem of total probability and the partition given by the various values of N, we have

$$\begin{aligned} r_C(y) &= P[Y > y] \\ &= \sum_{n=1}^{\infty} P[Y > y\,|\,N=n]P[N=n] \\ &= \sum_{n=1}^{\infty} \exp\{-n\lambda y^{\alpha}\}\,\theta(1-\theta)^{n-1} \\ &= \frac{\theta}{1-\theta}\sum_{n=1}^{\infty} \left(\exp\{-\lambda y^{\alpha}\}(1-\theta)\right)^n \\ &= \frac{\theta\exp\{-\lambda y^{\alpha}\}}{1-\exp\{-\lambda y^{\alpha}\}(1-\theta)} && \text{(summing a geometric progression)} \end{aligned}$$

[4 MARKS]

and hence $r_C(y) \to 0$ as $y \to \infty$. This is intuitively and mathematically reasonable, as the chain is bound to break eventually as there is at least one defective link.

[1 MARK]

2.We have, in the with replacement case, for integer values of x satisfying $1 \leq x \leq n$

$$P[X \leq x] = P(r \text{ values not greater than } x) = \prod_{i=1}^{r} P(\text{value not greater than } x \text{ on draw } i) = \left(\frac{x}{n}\right)^r$$

by independence of draws. For non integer values of x,$P[X \leq x] = P[X \leq [x]]$ by definition Hence

$$\begin{aligned} F_X(x) &= \left(\frac{[x]}{n}\right)^r \\ f_X(x) &= \left(\frac{x}{n}\right)^r - \left(\frac{x-1}{n}\right)^r \qquad x = 1, 2, ..., n \end{aligned}$$

(marks for $F_X(x)$ only; $f_X(x)$ for exercise).

[*4 MARKS*]

Now, without replacement, we have for integer values of y that

$$P[Y \leq y] = \frac{\binom{y}{r}}{\binom{n}{r}}$$

as the numerator is the number of ways of choosing r numbers less than or equal to y, and the denominator is the number of ways of choosing r numbers less than or equal to n.

Hence

$$\begin{aligned} F_Y(y) &= \frac{\binom{[y]}{r}}{\binom{n}{r}} \\ f_Y(y) &= \frac{\binom{y}{r} - \binom{y-1}{r}}{\binom{n}{r}} = \frac{\binom{y-1}{r-1}}{\binom{n}{r}} \qquad y = 1, 2, ..., n \end{aligned}$$

(marks for $F_Y(y)$ only; $f_Y(y)$ for exercise).

[*4 MARKS*]

and for $1 \leq z < n$

$$F_Y(z) = \frac{z!(n-r)!}{(z-r)!n!} < \left(\frac{z}{n}\right)^r = F_X(z)$$

[*2 MARKS*]

Note: the calculation of F_X and F_Y can be achieved through many routes, for example by using conditional probability arguments for successive draws, or by using the result for the Maximum score of independent repeats of the same drawing experiment, or by consideration of the mass function directly. All of these methods are acceptable.