EXERCISES 4

Multivariate Transformations, Limits of Moment Generating Functions, Order Statistics

1. If X and Y are continuous random variables with joint pdf $f_{X,Y}$, with

$$f_{X,Y}(x,y) = c(\alpha,\beta)\ x^{\alpha-1}(1-x)^{\beta-1}\ \ c(\alpha+\beta,\gamma)\ y^{\alpha+\beta-1}(1-y)^{\gamma-1} \qquad 0 < x < 1,\ 0 < y < 1.$$

and zero otherwise, for $\alpha,\beta,\gamma > 0$, where normalizing constants $c(\alpha,\beta)$ and $c(\alpha+\beta,\gamma)$ are defined by

$$\frac{1}{c(a,b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\ dt$$

Let U be defined by $U = XY$. Recalling from lecture notes that

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}(x,u/x)|x|^{-1}\ dx$$

find the marginal pdf of random variable $U = XY$. Note that, as $\mathbb{X} = \mathbb{Y} = (0,1)$, we have $XY < X$.

2. Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = c\ xy(1-x-y) \qquad 0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1.$$

for some constant $c > 0$. Find the covariance of X and Y.

3. Suppose that X and Y have joint pdf that is *constant* on the range $\mathbb{X}^{(2)} = (0,1)\times(0,1)$, and zero otherwise. Find the marginal pdf of random variables $U = X/Y$ and $V = -\log(XY)$. stating clearly the range of the transformed random variable in each case. Find the pdf and cdf of $Z = X - Y$.

4. Suppose that continuous random variables X_1, X_2, X_3 are independent, and have marginal pdfs specified by

$$f_{X_i}(x_i) \ = \ c_i x_i^i\ \mathrm{e}^{-x_i} \qquad x_i > 0$$

for $i = 1,2,3$. Find the joint pdf of random variables Y_1, Y_2, Y_3 defined by

$$Y_1 \ = \ X_1/(X_1+X_2+X_3) \qquad Y_2 \ = \ X_2/(X_1+X_2+X_3) \qquad Y_3 \ = \ X_1+X_2+X_3$$

and evaluate the marginal expectation of Y_1.

5. Suppose that X and Y are continuous random variables with pdf given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(x^2+y^2\right)\right\} \qquad x,y\in\mathbb{R}$$

Let random variable U be defined by $U = X/Y$. Find the pdf of U.

Suppose now that S is a random variable, independent of X and Y, with pdf given by

$$f_S(s) = c(\nu)\ s^{\nu/2-1}e^{-s/2} \qquad s > 0$$

where ν is a positive integer and $c(\nu)$ is a constant function of ν. Find the pdf of random variable T defined by $T = X/\sqrt{S/\nu}$.

6. Suppose that the joint pdf of random variables X and Y is specified via the conditional density $f_{X|Y}$ and the marginal density f_Y as

$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \qquad x \in \mathbb{R} \qquad f_Y(y) = c(\nu) y^{\nu/2-1} e^{-\nu y/2} \qquad y > 0$$

where ν is a positive integer, Find the marginal pdf of X.

7. Suppose that discrete random variable X has mass function given by

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, ...$$

and zero otherwise, for $\lambda > 0$, and let Z_1 be defined by $Z_1 = (X - \lambda)/\sqrt{\lambda}$. Find the mgf of Z_1. and limiting mgf as as $\lambda \longrightarrow \infty$.

Now suppose that X is continuous, with pdf given by $f_X(x) = cx^{\lambda-1}e^{-x}$ for $x > 0$, and zero otherwise, for $\lambda > 0$. For Z_2 defined by $Z_2 = (X - \lambda)/\sqrt{\lambda}$, find the mgf of Z_2, and the limiting mgf as $\lambda \longrightarrow \infty$.

8. Suppose that X_1 and X_2 are independent continuous random variables with pdf f_X and cdf F_X. Let Y_1 and Y_2 be the smaller and larger of the pair $\{X_1, X_2\}$ respectively. If constant c is defined by $F_X(c) = p$, for some p where $0 \le p \le 1$, evaluate $\mathrm{P}[Y_2 > c]$.

9. Suppose that $X_1, ..., X_k$ are i.i.d continuous random variables with density function

$$f_X(x) \;\; = \;\; \lambda e^{-\lambda x} \quad x > 0$$

and zero otherwise, for some $\lambda > 0$. Find the marginal pdf of the minimum order statistic, Y_1.

Now suppose that $X_1, ..., X_k$ are i.i.d continuous random variables with density function

$$f_X(x) \;\; = \;\; 1/x^2 \qquad 1 \le x < \infty$$

Find the joint pdf of the order statistics $Y_1, ..., Y_k$ and the marginal pdf of the smallest order and largest order statistics Y_1 and Y_k.

10. Suppose that $X_1, ...X_k$ are i.i.d. continuous random variables with pdf that is constant on $(0, 1)$, zero otherwise, and let Y_k be the corresponding largest order statistic. Find the smallest k such that $\mathrm{P}[Y_k \ge 0.99] \ge 0.95$.

11. Suppose that X is a continuous random variable with pdf given by

$$f_X(x) = 5x^4 \qquad 0 \le x \le 1$$

and zero otherwise. Suppose that f_X above is the pdf for the marks (out of 100, and rescaled to 0-1) for a given student in a test. If test results for k tests are regarded as i.i.d. random variables having pdf f_X, let Y_1 be the random variable corresponding to the minimum score obtained in the k tests, evaluate $\mathrm{P}[Y_1 \le 0.75]$ if $k = 3, 4, 5$. Find the pdf of Y_1, and the limiting probability distribution of Y_1 as the number of tests, k becomes large.

12. Suppose that $X_1, ..., X_n$ are are iid from pdf given by

$$f_X(x) = \frac{(\alpha+1)x^\alpha}{\theta^{\alpha+1}} \quad 0 \le x \le \theta$$

and zero otherwise, for parameters $\alpha, \theta > 0$. Let $Y_1, ..., Y_n$ be the corresponding order statistics derived from $X_1, ..., X_n$. By considering cdfs, find the distributions of Y_1 and Y_n as $n \longrightarrow \infty$.