M2S1 - EXERCISES 3

Moment Generating Functions / Discrete and Continuous Multivariate Distributions

1. The continuous random variable X with range $\mathbb{X} = \mathbb{R}$ has pdf given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} \qquad x \in \mathbb{R}$$

Suppose that $X_1, ..., X_k$ are independent random variables having the same distribution as X.

Find the mgf of random variable X, the pdf and the mgf of random variable Y

$$Y = \mu + \sigma X.$$

for parameters μ and σ. Now let random variable T be defined by $T = X^2$. Find the pdf and mgf of T.

2. Suppose that random variable X has mgf given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}$$

Find the probability distribution, and the expectation and variance of X.

3. Suppose that random variable X has mgf given by

$$M_X(t) = \left(1 - \theta + \theta e^t\right)^n$$

for some θ, where $0 \leq \theta \leq 1$. Obtain a power series expansion for $M_X(t)$, and hence identify $\mathrm{E}_{f_X}[X^r]$ for $r = 1, 2, 3, ...$

4. Suppose that X is a continuous random variable with pdf

$$f_X(x) \;=\; \exp\{-(x+2)\} \qquad -2 < x < \infty$$

Find the mgf of X, and hence find the expectation and variance of X.

5. Suppose that X is a random variable with mass function/pdf f_X and mgf M_X. The *cumulant generating function* of X, K_X, is defined by $K_X(t) = \log[M_X(t)]$. Prove that

$$\frac{d}{dt}\left\{K_X(t)\right\}_{t=0} = \mathrm{E}_{f_X}[X] \qquad\qquad \frac{d^2}{dt^2}\left\{K_X(t)\right\}_{t=0} = \mathrm{Var}_{f_X}[X]$$

6. Suppose that X and Y are discrete random variables with joint mass function given by

$$f_{X,Y}(x,y) = \;\; c\,\frac{2^{x+y}}{x!\;y!} \qquad x,\; y = 0, 1, 2, ...$$

and zero otherwise, for some constant c. Find the value of c, and the marginal mass functions of X and Y. Prove that X and Y are independent random variables.

7. Continuous random variables X and Y have joint cdf, $F_{X,Y}$ defined for $(x, y) \in \mathbb{R}^2$ by

$$F_{X,Y}(x, y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right) \qquad x \geq 0$$

and zero otherwise. Find the joint pdf, $f_{X,Y}$. Are X and Y independent ?

8. The joint pdf of continuous random variables X and Y is given by

$$f_{X,Y}(x, y) = cx^{\alpha-1} \qquad x > 0,\ 0 < y < \exp\{-\beta x^{\alpha}\}$$

for parameters $\alpha, \beta > 0$, and zero otherwise, for constant c. Find the marginal pdfs of X and Y.

9. Are the following functions valid continuous joint cdfs for random variables X and Y ?

$$F_1(x, y) = 1 - e^{-x-y} \qquad x, y \geq 0 \qquad\qquad F_2(x, y) = \begin{cases} 1 - e^{-x} - xe^{-y} & 0 \leq x \leq y \\ 1 - e^{-y} - ye^{-x} & 0 \leq y \leq x \end{cases}$$

10. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x, y) \ = \ \frac{1}{2x^2 y} \qquad 1 \leq x < \infty,\ 1/x \leq y \leq x$$

and zero otherwise. Derive the marginal pdf of X, the marginal pdf of Y, the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$. Calculate the marginal expectation of Y, $E_{f_Y}[Y]$.

11. Suppose that X, Y and Z are continuous random variables with joint pdf given by

$$f_{X,Y,Z}(x, y, z) = \ c \qquad 0 < x < y < z < 1$$

and zero otherwise, for some constant c. Find the value of c, and

(a) the joint pdf of X and Z
(b) the joint pdf of Y and Z
(c) the conditional pdf of Y given $X = x$ and $Z = z$.
(d) the conditional pdf of X given $Y = y$ and $Z = z$.
(e) the joint conditional pdf of X and Y given $Z = z$.

12. For random variables X and Y with joint mass function/pdf $f_{X,Y}$, the **joint moment generating function** (joint mgf) is denoted $M_{X,Y}$, and is defined by

$$M_{X,Y}(t, u) = \mathrm{E}_{f_{X,Y}}\left[e^{tX+uY}\right].$$

Show that $M_X(t) = M_{X,Y}(t, 0)$ and $M_Y(u) = M_{X,Y}(0, u)$, and prove that for $j, k \geq 1$,

$$\mathrm{E}_{f_{X,Y}}\left[X^j Y^k\right] = \frac{\partial^{j+k}}{\partial t^j \partial u^k}\left\{M_{X,Y}(t, u)\right\}_{t=0,u=0}.$$

Hence find an expression for $\mathrm{Cov}_{f_{X,Y}}[X, Y]$ in terms of $M_{X,Y}$.