M2S1 - EXERCISES 2

*Questions marked † are elementary. Questions marked * are challenging. Unmarked questions are standard..*

1†. Show that the function, F_X, defined for $x \in \mathbb{R}$ by

$$F_X(x) = c \exp\left\{-e^{-x}\right\}$$

is a valid cdf for a continuous random variable X for a specific choice of constant c, and find the pdf, f_X associated with this cdf.

Now consider the function $g(x)$ defined by

$$g(x) = \frac{|x|}{(1+x^2)^2} \qquad x \in \mathbb{R}$$

Show that the function $f_X(x) = cg(x)$ (for some constant $c > 0$) is a valid pdf for a continuous random variable X with range $\mathbb{X} = \mathbb{R}$, and find the cdf, F_X, associated with this pdf, and the expected value of X, $\mathrm{E}_{f_X}[X]$.

2. The failure time of an electronic component is a continuous random variable X with cdf F_X defined by

$$F_X(x) = 1 - \frac{1}{(1+\lambda x)^2} \qquad x > 0$$

for parameter $\lambda > 0$, and $F_X(x) = 0$ for $x \leq 0$.

Find the pdf, f_X, and the expected value, $\mathrm{E}_{f_X}[X]$ of X. Find and interpret the conditional probability $\mathrm{P}[\ X > c_2 \mid X > c_1\ ]$ for constants $c_1 < c_2$.

Now suppose that, due to a fusing fault, there is a probability π $(0 \leq \pi < 1)$ that the component fails instantaneously (so that $X = 0$). Find the cdf of X, and show that random variable X is only continuous (according to the definition given in lectures) if $\pi = 0$.

Write down and then evaluate a reasonable definition for the expectation of X in terms of π, F_X and λ. Justify the definition you use.

3. Let X be a continuous random variable with range $\mathbb{X} = \mathbb{R}^+$, pdf f_X and cdf F_X. By writing the expectation in its integral definition form on the left hand side, and changing the order of integration show that

$$\mathrm{E}_{f_X}[X] = \int_0^\infty [1 - F_X(x)]\ dx$$

Using an identical approach, show also that for integer $r \geq 1$,

$$\mathrm{E}_{f_X}[X^r] = \int_0^\infty r x^{r-1} [1 - F_X(x)]\ dx$$

N.B. Take care with the integration...

4*. Suppose that continuous random variables X_1 and X_2 both with range $\mathbb{X} = \mathbb{R}^+$ have pdfs f_1 and f_2 respectively such that

$$f_1(x) = cx^{-1} \exp\left\{-(\log(x))^2/2\right\} \qquad x > 0 \qquad\qquad f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right] \qquad x > 0$$

and $f_1(x) = f_2(x) = 0$ for $x \leq 0$. If, for $r = 1, 2, \ldots$, $\mathrm{E}_{f_1}[X_1^r] = \exp\left\{r^2/2\right\}$, show that

$$\mathrm{E}_{f_2}[X_2^r] = \exp\left\{r^2/2\right\}$$

Hint: write out the integral for $\mathrm{E}_{f_2}[X_2^r]$, and then make a transformation $t = \log(x)$ in the integral. Then complete the square.

5. Suppose that X is a continuous random variable with range $\mathbb{R}$ and pdf given by

$$f_X(x) = \alpha^2 x \exp\{-\alpha x\} \qquad x \geq 0$$

and zero otherwise, for parameter $\alpha > 0$.

(i) Find the cdf of X, F_X, and hence show that, for any positive value m,

$$\mathrm{P}[X \geq m] = (1 + \alpha m) \exp\{-\alpha m\}$$

(ii) Find $\mathrm{E}_{f_X}[X]$. If the expected value of X is increased to $2/\beta$ (for $0 < \beta < \alpha$), find the associated change in $\mathrm{P}[X \geq m]$.

6.[†] Suppose that X is a continuous random variable with density function given by

$$f_X(x) = 4x^3 \qquad 0 < x < 1$$

and zero otherwise. Find the density functions of the following random variables

$$\begin{array}{llll} \text{(a)} & Y = X^4 & \text{(b)} & W = e^X \\ \text{(c)} & Z = \ln X & \text{(d)} & U = (X - 0.5)^2 \end{array}$$

Find the monotonic decreasing function H such that the random variable V, defined by $V = H(X)$, has a density function that is constant on the interval $(0, 1)$, and zero otherwise.
Now suppose that X is a continuous random variable with density function given by

$$f_X(x) = 1 \qquad 0 < x < 1$$

and zero otherwise. Find the density functions of the following random variables

$$\begin{array}{llll} \text{(a)} & Y = X^{1/4} & \text{(b)} & W = e^{-X} \\ \text{(c)} & Z = 1 - e^{-X} & \text{(d)} & U = X(1 - X) \end{array}$$

7. The measured radius of a circle, R, is a continuous random variable with density function given by

$$f_R(r) = 6r(1 - r) \qquad 0 < r < 1$$

and zero otherwise. Find the density functions of the circumference and the area of the circle.

8. Suppose that X is a continuous random variable, and that the cdf of X, F_X, is a one-to-one function. Show that the random variable $Y = F_X(X)$ has a pdf that is constant on the interval $(0, 1)$, and zero elsewhere.

9.* Suppose that X is a continuous random variable with density function given by

$$f_X(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \qquad x > 0$$

for constants $\alpha, \beta > 0$, and zero otherwise. Find the density function and cdf of the random variable defined by $Y = \ln X$, and the density function of the random variable defined by $Z = \xi + \theta Y$.

10*. Continuous random variable X has range (a, b), and pdf given by

$$f_X(x) = k(x - a)(b - x) \qquad a < x < b$$

and zero otherwise, for some constant k.

(i) Sketch f_X, and compute the value of k.
(ii) Find the pdf of random variable Y defined by $Y = (X - a)/(b - a)$.
(iii) Find the pdf of random variable Z defined by $Z = Y(1 - Y)$.