M2S1 - ASSESSED COURSEWORK 3

To be handed in no later than Thursday, 14th December

1. (a) By evaluating its mgf, M_Y, identify the distribution of random variable Y defined in terms of independent random variables X_1 and X_2 by

$$Y = X_1 + X_2$$

when

(i) $X_1 \sim Binomial(n_1, \theta)$ and $X_2 \sim Binomial(n_2, \theta)$

(ii) $X_1 \sim NegBinomial(n_1, \theta)$ and $X_2 \sim NegBinomial(n_2, \theta)$

(iii) $X_1 \sim Exponential(\lambda)$ and $X_2 \sim Exponential(\lambda)$

(iv) $X_1 \sim Gamma(\alpha_1, \beta)$ and $X_2 \sim Gamma(\alpha_2, \beta)$

(v) $X_1 \sim Normal(\mu_1, \sigma_1^2)$ and $X_2 \sim Normal(\mu_2, \sigma_2^2)$

(b) Suppose now that random variables X and Y and joint probability distribution specified as

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) \qquad x \in \{0, 1, ...\}, y \in \mathbb{R}^+$$

by

$$f_{X|Y}(x|y) = \frac{e^{-y} y^x}{x!} \qquad x \in \{0, 1, ...\} \qquad\qquad f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \qquad y \in \mathbb{R}^+$$

where α is a positive integer, so that

$$X|Y = y \sim Poisson(y) \qquad\qquad Y \sim Gamma(a, \beta).$$

By evaluating

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y) \, dy$$

show that, marginally, X has a Negative Binomial distribution.

Write down the expectation and variance of X in terms of α and β.

2. (a) Suppose that X is a continuous random variable with range $\mathbb{X}$ and pdf f_X, and suppose that

$$\mathrm{E}_{f_X}[X] = \mu \qquad\qquad \mathrm{Var}_{f_X}[X] = \sigma^2.$$

Suppose also that g is a *non-negative* real function, and that $c > 0$ is a constant. Let $\mathcal{A}$ be the set defined by

$$\mathcal{A} = \{x \in \mathbb{R} : g(x) \geq c\}.$$

Consider the expectation of the function $g(X)$ with respect to f_X. By writing

$$\mathrm{E}_{f_X}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{\mathcal{A}} g(x) f_X(x)\, dx + \int_{\mathcal{A}'} g(x) f_X(x)\, dx$$

and noting that $g(x) \geq c$ for $x \in \mathcal{A}$, show that

$$\mathrm{E}_{f_X}[g(X)] \geq c\,\mathrm{P}[\, X \in \mathcal{A}\,] = c\,\mathrm{P}[\, g(X) \geq c\,]$$

(b) By setting $g(X) = (X-\mu)^2$ and $c = k^2\sigma^2$ for $k > 0$ in (a), show that

$$\mathrm{P}[\, |X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

(c) Suppose finally that $X_1, ..., X_n$ are independent and identically distributed random variables, with properties identical to X. Let random variable Y_n be defined by

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Show, using the general peoperties of expectations, that

$$\mathrm{E}_{f_{Y_n}}[Y_n] = \mu \qquad\qquad \mathrm{Var}_{f_{Y_n}}[Y_n] = \frac{\sigma^2}{n}.$$

and hence deduce from (b) that, for any $\epsilon > 0$,

$$\mathrm{P}[\, |Y_n-\mu| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2}$$

Explain the implications of this result as $n \to \infty$.