M2S1 - ASSESSED COURSEWORK 1

To be handed in no later than Monday, 30th October

1. The number, N, of defective links in a chain, is a discrete random variable whose probability distribution is defined by the model

$$P[N=n]=\theta(1-\theta)^{n-1} \qquad n=1,2,...$$

for some $0<\theta<1$, that is, it is known that there will be at least one defective link, but the actual number is unknown. The defective links are prone to breaking; the probability that a defective link breaks earlier than time y after construction is specified by the function

$$F_L(y)=1-e^{-\lambda y^{\alpha}}$$

for $y>0$, where α and λ are real, positive constants. Defective links break (mutually) independently of each other; non-defective links never break.

By considering the reliability function

$$r_L(y)=P(\text{Defective link breaks later than time } y)$$

show that, if the number of defective links was known to be n, then the probability that the chain is broken earlier than time y is

$$1-\exp\{-n\lambda y^{\alpha}\}$$

and hence find the reliability function for the chain, $r_C(y)$ for $y>0$ if N is not observed.
Show that

$$\lim_{y\longrightarrow\infty} r_C(y)=0$$

and comment on this result.

2. Lottery tickets numbered 1,2,...n are placed in a hat and r are to be drawn out. Two mechanisms are to be used. Let X be the discrete random variable representing the largest number drawn when the tickets are *replaced* after each single draw, and let Y be the random variable representing the largest number drawn when the tickets are *not replaced*.

Show that the cumulative distribution functions (cdfs) of X and Y are, respectively

$$F_X(x)=\left(\frac{[x]}{n}\right)^r \quad 0\le x\le n \qquad\qquad F_Y(y)=\frac{\binom{[y]}{r}}{\binom{n}{r}} \quad 0\le y\le n$$

where $[t]$ denotes the largest integer not greater than t.

Verify that for all z

$$F_Y(z)\le F_X(z).$$

Hint: the number of ways of selecting m items from M available with replacement is m^M; without replacement the number is $\binom{M}{m}$.

[Exercise for no extra marks: find the probability mass functions of X and Y.]

M2S1 - SUPPLEMENTARY 1

1. For which values of the constant c do the following functions define valid probability mass functions for a discrete random variable X, taking values on range $\mathbb{X} = \{1, 2, 3, ...\}$:

(a) $f_X(x) = c/2^x$ (b) $f_X(x) = c/(x2^x)$

(c) $f_X(x) = c/(x^2)$ (d) $f_X(x) = c2^x/x!$

In each case, calculate (where possible) $\mathrm{P}[X > 1]$ and $\mathrm{P}[X$ is even$]$

2. n identical fair coins are tossed. Those that show Heads are tossed again, and the number of Heads obtained on the second set of tosses defines a discrete random variable X. Assuming that all tosses are independent, find the range and probability mass function of X.

3. A point is to be selected from an integer lattice restricted to the triangle with corners at $(1, 1)$, $(n, 1)$ and (n, n) for positive integer n. If all points are equally likely to be selected, find the probability mass functions for the two discrete random variables X and Y corresponding to the $x-$ and $y-$ coordinates of the selected points respectively.

4. A continuous random variable X has pdf given by

$$f_X(x) = c(1-x)x^2 \qquad 0 < x < 1$$

and zero otherwise. Find the value of c, the cdf of X, F_X, and $\mathrm{P}[X > 1/2]$.

5. A function f is defined by

$$f(x) = k/x^{k+1} \qquad x > 1$$

and zero otherwise. For what values of k is f a valid pdf ? Find the cdf of X.

6. A continuous random variable X has pdf given by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and zero otherwise. Sketch f_X, and find the cdf F_X.

7. A continuous random variable X has cdf given by

$$F_X(x) = c(\alpha x^\beta - \beta x^\alpha) \qquad 0 \leq x \leq 1$$

for constants $1 \leq \beta < \alpha$, and zero otherwise.

Find the value of constant c, and evaluate the rth moment of X.

8. A continuous random variable X has cdf given by

$$F_X(x) = \frac{2\beta x}{\beta^2 + x^2} \qquad 0 \leq x \leq \beta$$

for constant $\beta > 0$. Find the pdf of X, and show that the expectation of X is

$$\beta(1 - \log 2)$$