M2S1 : EXERCISE SHEET 3 : SOLUTIONS

1. We have $\mathbb{X} = \{0, 1, 2\}$, and $\mathbb{Y} = \{0, 1\}$ by definition. Then joint mass function $f_{X,Y}$ is given by

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x) = \mathrm{P}[\ X = x\ ]\mathrm{P}[\ Y = y \mid X = x\ ]$$

where

$$\begin{aligned} \mathrm{P}[\ X = x\ ] &= \binom{2}{x}\left(\frac{1}{2}\right)^2 \qquad x = 0, 1, 2 \\ \mathrm{P}[\ Y = y \mid X = x\ ] &= \begin{cases} 1 & x = 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence the joint mass function is specified by the following table

		x = 0	1	2
y	0	$\frac{1}{4}$	$\frac{1}{2}$	0
	1	0	0	$\frac{1}{4}$

2. We have $X \sim Bernoulli(\theta)$, so that $f_X(x) = \theta^x(1-\theta)^{1-x}$ for $x = 0, 1$, and hence

$$\mathrm{P}[\ Y = y\ ] = \mathrm{P}[\ 1 - X = y\ ] = \mathrm{P}[\ X = 1 - y\ ] = \theta^{1-y}(1-\theta)^y$$

for $y = 0, 1$, so $Y \sim Bernoulli(1-\theta)$. Then

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x) = \mathrm{P}[\ X = x\ ]\mathrm{P}[\ Y = y \mid X = x\ ] = \begin{cases} \theta & x = 1, y = 0 \\ 1-\theta & x = 0, y = 1 \end{cases}$$

Similarly

$$f_{X,Z}(x, z) = f_X(x) f_{Z|X}(z|x) = \mathrm{P}[\ X = x\ ]\mathrm{P}[\ Z = z \mid X = x\ ] = \begin{cases} 1-\theta & x = 0, z = 0 \\ \theta & x = 1, z = 0 \end{cases}$$

3. $F_{X,Y}$ is continuous and non decreasing in x and y, and

$$\lim_{x \longrightarrow -\infty} F_{X,Y}(x, y) = \lim_{y \longrightarrow -\infty} F_{X,Y}(x, y) = 0 \qquad \lim_{x,y \longrightarrow \infty} F_{X,Y}(x, y) = 1$$

so $F_{X,Y}$ is a valid cdf, and

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial t_1 \partial t_2}\{F_{X,Y}(t_1, t_2)\}_{t_1 = x, t_2 = y} = \frac{e^{-x}}{\pi(1+y^2)} = f_X(x) f_Y(y)$$

so as $\mathbb{X}^{(2)} = \mathbb{R}^+ \times \mathbb{R}$, X and Y are independent.

Note: we carry out **partial** (rather than ordinary) differentiation of $F_{X,Y}$ with respect to x and y in order to calculate $f_{X,Y}$; there is a mistake in the lecture notes, p. 26, formula for the joint pdf in terms of the joint cdf. Apologies.

4. The form of the joint range $\mathbb{X}^{(2)}$ is the key point; we have $\mathbb{X}^{(2)} = \{\ (x,y)\ :\ x > 0, 0 < y < \exp\{-\beta x^{\alpha}\}\}$, and hence

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy \quad = \int_0^{e^{-\beta x^{\alpha}}} c x^{\alpha-1}\ dy = c x^{\alpha-1} \exp\{-\beta x^{\alpha}\} \qquad x > 0$$

$$\Longrightarrow F_X(x) \quad = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x c t^{\alpha-1} \exp\{-\beta t^{\alpha}\} = \frac{c}{\alpha\beta}\left(1 - \exp\{-\beta x^{\alpha}\}\right)$$

so that $c = \alpha\beta$. Similarly, letting $g(y) = \{-\log y/\beta\}^{1/\alpha}$, we have $0 < x < g(y)$ as $0 < y < \exp\{-\beta x^{\alpha}\}$, and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^{g(y)} c x^{\alpha-1}\ dx = \frac{c}{\alpha}\{g(y)\}^{\alpha} = -\log y \qquad 0 < y < 1$$

5. (a) We have

$$\frac{\partial^2}{\partial t_1 \partial t_2}\{F_1(t_1,t_2)\}_{t_1=x, t_2=y} = -e^{-x-y} < 0$$

on the specified range of X and Y, so $F_{X,Y}$ is not a valid cdf, as this partial derivative must be non-negative.

(b) We have

$$\frac{\partial^2}{\partial t_1 \partial t_2}\{F_2(t_1,t_2)\}_{t_1=x, t_2=y} = \begin{cases} e^{-y} & 0 \le x \le y \\ \\ e^{-x} & 0 \le y \le x \end{cases}$$

which is non-negative everywhere. Note that, $F_2(x,0) = F_2(0,y) = 0$. However, consider the behaviour of F_2 as x and y become large; first, consider the cdf F defined by

$$F(x,y) = 1 - e^{-y} - y e^{-x} \qquad 0 \le y \le x < \infty,$$

that is, identical to F_2 on only half the original domain. It is easy to check that F is a cdf, in particular, that

$$\lim_{x,y \longrightarrow \infty} F(x,y) = 1$$

Hence, by symmetry we must have that

$$\lim_{x,y \longrightarrow \infty} F_2(x,y) = 2,$$

so F_2 is not a valid cdf.

The question as given contained a typo; the cdf F_2 was meant to be defined as

$$F_2(x,y) = \begin{cases} 1 - e^{-x} - x e^{-y} & 0 \le x \le y \\ \\ 1 - e^{-y} - y e^{-y} & 0 \le y \le x \end{cases}$$

in which case we have

$$f_2(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\{F_2(t_1,t_2)\}_{t_1=x, t_2=y} = e^{-y} \qquad 0 \le x \le y < \infty$$

and zero otherwise, which is also non-negative everywhere. Again, $F_2(x,0) = F_2(0,y) = 0$, and here

$$\lim_{x,y \longrightarrow \infty} F_2(x,y) = 1$$

so F_2 is a valid cdf.

Note also that, in the corrected question the marginal cdfs for X and Y are given by

$$F_X(x) \quad = \lim_{y \longrightarrow \infty} F_2(x,y) = 1 - e^{-x} \qquad x \ge 0$$

$$F_Y(y) \quad = \lim_{x \longrightarrow \infty} F_2(x,y) = 1 - (y+1)e^{-y} \qquad y \ge 0$$

6.(i) Put $U = X/Y$ and $Z = X$; the inverse transformations are therefore $X = Z$ and $Y = Z/U$, and note that the new variables are constrained by $0 < Z < \min\{U, 1\}$, as $Y < 1$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$g_1(t_1,t_2) = t_1/t_2 \qquad g_1^{-1}(t_1,t_2) = t_2$$

$$g_2(t_1,t_2) = t_1 \qquad g_2^{-1}(t_1,t_2) = t_2/t_1$$

and the Jacobian of the transformation is given by

$$J(u,z) = \begin{vmatrix} 0 & 1 \\ -z/u^2 & 1/u \end{vmatrix} = \frac{z}{u^2}$$

and hence

$$f_{U,Z}(u,z) = f_{X,Y}(z, z/u)\ z/u^2 = z/u^2 \qquad (u,z) \in \mathbb{U}^{(2)} = \{(u,z) : 0 < z < \min\{u,1\}, u > 0\}$$

and zero otherwise, and so

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u,z)\ dz = \int_0^{\min\{u,1\}} z/u^2\ dz = \frac{(\min\{u,1\})^2}{2u^2} \qquad u > 0.$$

(ii) Put $V = -\log(XY)$ and $Z = -\log X$; the inverse transformations are therefore $X = e^{-Z}$ and $Y = e^{-(v-z)}$, and note that $0 < Z < V$. In terms of the theorem, we have transformation functions defined by

$$g_1(t_1,t_2) = -\log(t_1 t_2) \qquad g_1^{-1}(t_1,t_2) = e^{-t_2}$$

$$g_2(t_1,t_2) = -\log t_1 \qquad g_2^{-1}(t_1,t_2) = e^{-(t_1-t_2)}$$

and the Jacobian of the transformation is given by

$$J(v,z) = \begin{vmatrix} 0 & -e^{-z} \\ -e^{-(v-z)} & e^{-(v-z)} \end{vmatrix} = e^{-v}$$

and hence

$$f_{V,Z}(v,z) = f_{X,Y}(e^{-z}, e^{-(v-z)})\ e^{-v} = e^{-v} \qquad (v,z) \in \mathbb{V}^{(2)} = \{(v,z) : 0 < z < v < \infty\}$$

and zero otherwise, and so

$$f_V(v) = \int_{-\infty}^{\infty} f_{V,Z}(v,z)\ dz = \int_0^v e^{-v}\ dz = ve^{-v} \qquad v > 0$$

and zero otherwise. Note that we could attempt (i) and (ii) together by setting

$$\begin{array}{l} U = X/Y \\ V = -\log(XY) \end{array} \iff \begin{array}{l} X = U^{1/2}e^{-V/2} \\ Y = U^{-1/2}e^{-V/2} \end{array}$$

note that, as X and Y lie in $(0,1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$J(u,v) = \begin{vmatrix} \dfrac{u^{-1/2}e^{-v/2}}{2} & -\dfrac{u^{1/2}e^{-v/2}}{2} \\ -\dfrac{u^{-3/2}e^{-v/2}}{2} & -\dfrac{u^{-1/2}e^{-v/2}}{2} \end{vmatrix} = u^{-1}e^{-v}/2.$$

Hence

$$f_{U,V}(u,v) = u^{-1}e^{-v}/2 \qquad 0 < e^{-v} < \min\{u, 1/u\},\ u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\ dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u}\ dv = \left[-\frac{e^{-v}}{2u}\right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \qquad u > 0 \\
f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\ du = \int_{e^{-v}}^{e^{v}} \frac{e^{-v}}{2u}\ du = \left[\frac{\log u}{2} e^{-v}\right]_{e^{-v}}^{e^{v}} = ve^{-v} \qquad v > 0
\end{aligned}$$

7. Easy to see that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, with $\mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$, so X and Y are independent, where

$$f_X(x) = \sqrt{c}\, \frac{2^x}{x!} \qquad f_Y(y) = \sqrt{c}\, \frac{2^y}{y!} \qquad \text{and} \qquad \sum_{x=0}^{\infty} f_X(x) = 1 \Longrightarrow \sqrt{c} = e^{-2}$$

as the summation is identical to the power series expansion of e^z at $z = 2$ if $\sqrt{c} = e^{-2}$.

8. Find c by integrating over the support of the density:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dydx = \int_0^1\int_x^1 c(x+y)\ dydx = c\int_0^1 \left[xy + y^2/2\right]_x^1\ dx = c\int_0^1 (x + 1/2 - 3x^2/2)\ dx = \frac{c}{2}$$

so $c = 2$. The marginal pdfs are

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_x^1 2(x+y)\ dy = \left[2xy + y^2\right]_x^1 = 1 + 2x - 3x^2 \qquad 0 \le x \le 1 \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^y 2(x+y)\ dy = \left[x^2 + 2xy\right]_0^y = 3y^2 \qquad 0 \le y \le 1
\end{aligned}$$

Cdf: In general, we have

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\ dsdt$$

(i) If $x, y < 0$, $F_{X,Y}(x,y) = 0$ as pdf is zero whenever x or y is negative.

(ii) If $0 \le x \le y \le 1$, then

$$F_{X,Y}(x,y) = \int_0^x\int_t^y 2(t+s)\ dsdt = \int_0^x \left[2ts + s^2\right]_t^y dt = \int_0^x (2ty + y^2 - 3t^2)dt = \left[t^2y + ty^2 - t^3\right]_0^x = x^2y + xy^2 - x^3$$

(iii) If $0 \le y \le 1$, but $x > y$ then

$$F_{X,Y}(x,y) = \int_0^y\int_t^y 2(t+s)\ dsdt = \int_0^y \left[2ts + s^2\right]_t^y dt = \int_0^y (2ty + y^2 - 2t^2 - t^2)dt = \left[t^2y + ty^2 - t^3\right]_0^y = y^3$$

(iv) If $0 \le x \le 1$, but $y > 1$ then

$$F_{X,Y}(x,y) = \int_0^x\int_t^1 2(t+s)\ dsdt = \int_0^x \left[2ts + s^2\right]_t^1 dt = \int_0^x (2t + 1 - 2t^2 - t^2)dt = \left[t^2 + t - t^3\right]_0^x = x^2 + x - x^3$$

(v) If $x, y > 1$, then $F_{X,Y}(x,y) = 1$.

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2(x+y)}{3y^2} \quad 0 \le x \le y \qquad f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2(x+y)}{(1+3x)(1-x)} \quad x \le y \le 1$$

9. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$. The marginals are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_{1/x}^{x} \frac{1}{2x^2 y}\ dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\ dy = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\ dy = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

Marginal expectation of Y;

$$\mathrm{E}_{f_Y}[\ Y\ ] = \int_{-\infty}^{\infty} y f_Y(y)\ dy = \int_0^1 \frac{y}{2}\ dy + \int_1^{\infty} \frac{1}{2y}\ dy = \infty$$

as the second integral is divergent.

10. To compute c,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dxdydz = \int_0^1\int_0^z\int_0^y c\ dxdydz = c\int_0^1\int_0^z y\ dydz = c\int_0^1 z^2/2\ dz = c/6$$

so $c = 6$.

$$f_{X,Z}(x,z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy = \int_x^z 6\ dy = 6(z-x) \qquad 0 < x < z < 1$$

$$f_{Y,Z}(y,z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dx = \int_0^y 6\ dx = 6y \qquad 0 < y < z < 1$$

$$f_{Y|X,Z}(y|x,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Z}(x,z)} = \frac{1}{z-x} \qquad x < y < z$$

$$f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} = \frac{1}{y} \qquad 0 < x < y$$

$$f_{X,Y|Z}(x,y|z) = \frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)} = \frac{2}{z^2} \qquad x < y < z$$

as

$$f_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\ dy\ dx = \int_0^z\int_0^y 6\ dxdy = 3z^2$$

11. Let

$$Y = X_1 + X_2 \qquad\qquad X_1 = \frac{Y_1 + Y_2}{2}$$
$$\Longleftrightarrow$$
$$Z = X_1 - X_2 \qquad\qquad X_2 = \frac{Y_1 - Y_2}{2}$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square A with corners at $(0,0)$, $(1,1)$, $(0,2)$ and $(1,-1)$ bounded by the lines $z = 1+y$, $z = 1-y$, $z = -1+y$ and $z = -1-y$. Then

$$f_{Y,Z}(y,z) = \frac{1}{2} \qquad (y,z) \in A$$

and zero otherwise (hint: sketch the square A). Hence

$$f_Z(z) \quad = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\, dy = \begin{cases} \displaystyle\int_{-z}^{2+z} \frac{1}{2}\, dy \quad = 1+z & \quad -1 < z \le 0 \\[2ex] \displaystyle\int_{z}^{2-z} \frac{1}{2}\, dy \quad = 1-z & \quad 0 < z < 1 \end{cases}$$

12. The transformations are

$$Y_1 = \frac{X_1}{X_1+X_2+X_3} \qquad\qquad X_1 = Y_1 Y_3$$
$$Y_2 = \frac{X_1}{X_1+X_2+X_3} \quad \Longleftrightarrow \quad X_2 = Y_2 Y_3$$
$$\qquad\qquad\qquad\qquad X_3 = Y_3(1-Y_1-Y_2)$$
$$Y_3 = X_1 + X_2 + X_3$$

which gives Jacobian

$$J(y_1,y_2,y_3) = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & (1-y_1-y_2) \end{vmatrix} = y_3^2$$

Hence the joint pdf is given by

$$\begin{aligned} f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) &= f_{X_1,X_2,X_3}(y_1y_3, y_2y_3, y_3(1-y_1-y_2))J(y_1,y_2,y_3) \\ &= c_1 y_1 y_3 \exp\{-y_1y_3\}\ c_2 y_2^2 y_3^2 \exp\{-y_2y_3\}\ c_3 y_3^3(1-y_1-y_2)^3 \exp\{-y_3(1-y_1-y_2)\}\ y_3^2 \\ &= c_1c_2c_3 y_1 y_2^2 (1-y_1-y_2)^3\ y_3^8 \exp\{-y_3\} = f_{Y_1,Y_2}(y_1,y_2) f_{Y_3}(y_3) \end{aligned}$$

where

$$f_{Y_1,Y_2}(y_1,y_2) \propto y_1 y_2^2 (1-y_1-y_2)^3. \quad \text{and} \quad f_{Y_3}(y_3) \propto y_3^8 \exp\{-y_3\}$$

Hence $Y_3 \sim Gamma(9,1)$; the transformations give the constraints $0 < Y_1, Y_2 < 1$ and $0 < Y_1 + Y_2 < 1$, and $Y_3 > 0$. Now

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_2 = \int_0^{1-y_1} c y_1 y_2^2 (1-y_1-y_2)^3\, dy_2 = c y_1 (1-y_1)^6 \int_0^1 t^2(1-t)^3\, dt \quad (t = y_2/(1-y_1))$$

and hence

$$f_{Y_1}(y_1) \propto y_1(1-y_1)^6 \Longrightarrow Y_1 \sim Beta(2,7),\ f_{Y_1}(y_1) = 336 y_1 (1-y_1)^6 \qquad 0 < y_1 < 1$$

and hence

$$\mathrm{E}_{f_{Y_1}}[\, Y_1 \,] = \frac{2}{2+7} = \frac{2}{9}$$

as the expectation of a $Beta(\alpha,\beta)$ distribution is $\alpha/(\alpha+\beta)$ from notes.