M2S1 : EXERCISE SHEET 1 : SOLUTIONS

1. For events $A, B \subseteq \Omega$,

$$\omega \in A \cap B \Longleftrightarrow \omega \in A \text{ and } \omega \in B \qquad \omega \in A \cup B \Longleftrightarrow \omega \in A \text{ or } \omega \in B \text{ or } \omega \in A \cap B$$

Hence

(i) $\omega \in E \cap \emptyset \Longleftrightarrow \omega \in E$ and $\omega \in \emptyset$. No such ω exists $\therefore\ E \cap \emptyset = \emptyset$
$\omega \in E \cup \emptyset \Longleftrightarrow \omega \in E$ or $\omega \in \emptyset$ or $\omega \in E \cap \emptyset \Longleftrightarrow \omega \in E\ \therefore\ E \cup \Omega = E$ (as($\omega \in E \cap \emptyset = \emptyset$).

(ii) $\omega \in E \cap \Omega \Longleftrightarrow \omega \in E$ and $\omega \in \Omega \Longleftrightarrow \omega \in E\ \therefore\ E \cap \Omega = E$
$\omega \in E \cup \Omega \Longleftrightarrow \omega \in E$ or $\omega \in \Omega$ or $\omega \in E \cap \Omega \Longleftrightarrow \omega \in \Omega\ \therefore\ E \cup \Omega = \Omega$

(iii) $(E \cap F) \cup (E^{'} \cup F^{'}) = (\ (E \cap F) \cup E^{'}\ ) \cup (\ (E \cap F) \cup F^{'}\ )$
$= (\ (E \cup E^{'}) \cap (F \cup E^{'})\ ) \cup (\ (E \cup F^{'}) \cap (F \cup F^{'})\ )$
$= (\ \Omega \cap (F \cup E^{'})\ ) \cup (\ (E \cup F^{'}) \cap \Omega\ )$
$= (F \cup E^{'}) \cup (E \cup F^{'})$
$= \Omega$

$\therefore\ (E \cap F) \cup (E^{'} \cup F^{'}) = \Omega$ so $(E \cap F)^{'} = (E^{'} \cup F^{'})$ and $(E^{'} \cup F^{'})^{'} = (E \cap F)$

Also, $(E \cup F)^{'} = (E^{'} \cap F^{'})$ (by replacing E and F by $E^{'}$ and $F^{'}$)

(iv) $E \subseteq F \Longrightarrow F = E \cup G$, say, where $E \cap G = \emptyset \Longrightarrow F^{'} = (E \cup G)^{'} = E^{'} \cap G^{'} \subseteq E^{'}$

(v) If $E \subseteq F$, then $F = E \cup G$, say, where $E \cap G = \emptyset$ so

$E \cap F = E \cap (E \cup G) = (E \cap E) \cup (E \cap G) = E$
$E \cup F = E \cup (E \cup G) = (E \cup G) = F.$

2. An event is a subset of sample outcomes, and the number of subsets of a collection of k items is 2^k (each of the k outcomes can be either included or excluded from the subsets).

3. A collection of subsets, $\mathcal{A}$, of sample space Ω, say

$$\mathcal{A} = \{A_1, A_2, ...\},$$

is a σ-algebra if

(I) $\Omega \in \mathcal{A}$

(II) $A \in \mathcal{A} \Longrightarrow A^{'} \in \mathcal{A}$

(III) $A_1, A_2, ... \in \mathcal{A} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Taking these three conditions in turn we attempt to construct a minimal collection $\mathcal{A}_0$ of subsets that satisfies these conditions. Clearly (I) requires that $\Omega \in \mathcal{A}_0$, and (II) implies that, as $\Omega \in \mathcal{A}_0$, we require $\emptyset \in \mathcal{A}_0$. Now, if $\mathcal{A}_0 = \{\emptyset, \Omega\}$, then it can be verified that (III) holds as

$$\emptyset \cup \Omega = \Omega \in \mathcal{A}_0.$$

Hence $\mathcal{A}_0 = \{\emptyset, \Omega\}$ is the smallest σ-algebra of subsets of Ω.

4. If $\mathcal{A}$ is "countably additive" (or rather, closed under *countable* union), then we have

$$A_1, A_2, ... \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

for subsets $A_1, A_2, ...$ of Ω. But we also have $\emptyset \in \mathcal{A}$, so taking $A_j = \emptyset$ for $j = k+1, k+2, ...$, we have

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{k} A_i \qquad \implies \qquad \bigcup_{i=1}^{k} A_i \in \mathcal{A}$$

and hence $\mathcal{A}$ is "finitely additive" (or rather, closed under *finite* union).

5.(i) $\Omega = \{H, T\}$, so the set $\mathcal{A}$ of all subsets of Ω, is

$$\{\ \emptyset, \{H\}, \{T\}, \{H, T\} = \Omega\ \},$$

and it is easy to verify that $\mathcal{A}$ is a σ-algebra by checking conditions (I), (II) and (III).

(ii) If Ω is countable, say $\Omega = \{\ \omega_1, \omega_2, ..., \}$, let $\mathcal{A}$ be the set of all subsets of Ω. An event A_i is a subset of Ω if $\omega \in A_i \implies \omega \in \Omega$, so $\mathcal{A}$ is the set of events A_i that are countable unions of the elements of Ω.

Therefore

(I) $\Omega \in \mathcal{A}$ by construction.

(II) $A \in \mathcal{A} \implies A^{'} \in \Omega$ (as $A \cup A^{'} = \Omega$, and A and Ω are coutable unions of elements of Ω).

(III) $\mathcal{A}$ is closed under countable union (as the countable union of countable unions of elements of Ω is itself a coutable unions of elements of Ω).

and hence $\mathcal{A}$ is a σ-algebra.

6.(a) $\{A_k\}$ forms a partition of $\Omega = \mathbb{R}^+$ as $\omega \in \Omega \implies \omega \in A_k$ for precisely one k, and $\bigcup_{k=1}^{\infty} A_k = \Omega$.

Construct σ-algebra $\mathcal{A}$ whose elements are countable unions of the events A_k.

A countable union of elements of the $\{A_k\}$ sequence can be described by an infinite binary sequence in which the kth term is 1 if A_k is in the countable union, and 0 otherwise. It is then straightforward to verify that the conditions (I), (II) and (III).

(I) consider the infinite binary sequence whose elements are all 1s.

(II) If $A \in \mathcal{A}$, then $A^{'}$ has an binary sequence representation obtained by switching 1 to 0 and 0 to 1 for each element of the A sequence. Then clearly $A^{'} \in \mathcal{A}$.

(III) $\mathcal{A}$ is closed under countable union, by considering the binary sequence representation.

and hence $\mathcal{A}$ is a σ-algebra.

(b) Can construct all such intervals by taking unions and intersections of A_i events and their complements, (if we interpret $\{a_i\}$ as the interval $(a_i, a_i]$), as

$$A_i^{'} \cap A_j = (a_i, a_j] \qquad a_i < a_j$$

and

$$[a, b] = (a, b] \cup \{a\} \qquad [a, b) = [a, b] \cup \{b\} \qquad (a, b) = (a, b] \cap \{b\}^{'}$$

(ii) Straightforward to verify that conditions (I), (II), and (III) hold using the distributive and De Morgan laws