M2S1 - EXERCISES 4

In the following questions the following results may be used; if $X_1, ...X_n$ are a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with mass function/pdf f_X and cdf F_X, let Y_n and Z_n correspond to the maximum and minimum order statistics derived from $X_1, ...X_n$, that is

$$Y_n = \max\{X_1, ..., X_n\} \qquad Z_n = \min\{X_1, ..., X_n\}.$$

Then the cdfs of Y_n and Z_n are given by

$$F_{Y_n}(y) = \{F_X(y)\}^n \qquad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$

(you should check how to derive these results)

1. Suppose $X_1, ..., X_n \sim Uniform(0, 1)$, that is

$$F_X(x) = x \qquad 0 \leq x \leq 1$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

2. Suppose $X_1, ..., X_n \sim Exp(\lambda)$, that is

$$F_X(x) = 1 - e^{-\lambda x} \qquad x > 0$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

3. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{x} \qquad x \geq 1$$

Find the cdfs of Z_n and $U_n = Z_n^n$, and the limiting distributions of Z_n and U_n as $n \longrightarrow \infty$.

4. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}} \qquad x \in \mathbb{R}$$

Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \longrightarrow \infty$.

5. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x} \qquad x > 0$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

Find also the cdfs of

$$U_n = Y_n/n \qquad V_n = nZ_n$$

and the limiting distributions of U_n and V_n as $n \longrightarrow \infty$.

6. Suppose $X_1, ..., X_n \sim Exp(\lambda)$. Find the cdf of

$$U_n = \lambda Y_n - \log n$$

and the limiting distribution of U_n as $n \longrightarrow \infty$.

7. Suppose $X_1, ..., X_n \sim Bernoulli(\theta)$. Write down the mgf of X_i, the mgf of

$$S_n = \sum_{i=1}^{n} X_i$$

and the mgf of

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{S_n}{n}.$$

Find the limiting form of the mgf, and hence identify the limiting distribution of M_n as $n \longrightarrow \infty$. Explain the implications of this result for statistical parameter estimation.

8. Suppose $X_1, ..., X_n \sim Poisson(\lambda)$. Let

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Show that

$$M_n \xrightarrow{p} \lambda$$

as $n \longrightarrow \infty$. If random variable T_n is defined by

$$T_n = e^{-M_n}$$

show that

$$T_n \xrightarrow{p} e^{-\lambda}$$

Using the central limit theorem, find the approximate probability distribution of T_n as $n \longrightarrow \infty$.