M2S1 - EXERCISES 3

1. A fair coin is tossed twice with the outcomes independent. Let X be the random variable corresponding to the number of Heads obtained, and let Y be the random variable defined by

$$Y = \begin{cases} 1 & X = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint probability mass function for (X, Y).

2. Let X be the discrete (binary) random variable taking values 0 and 1 with probabilities $1 - \theta$ and θ respectively. Let random variables Y and Z be defined by

$$Y = 1 - X \qquad Z = XY$$

Find the joint mass function of X and Y, $f_{X,Y}$, and the joint mass function of X and Z, $f_{X,Z}$.

3. Continuous random variables X and Y have joint cdf, $F_{X,Y}$ defined for $(x, y) \in \mathbb{R}^2$ by

$$F_{X,Y}(x, y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right) \qquad x \geq 0$$

and zero otherwise. Verify that $F_{X,Y}$ satisfies the properties for a continuous cdf, and find the joint pdf, $f_{X,Y}$ corresponding to this cdf. Are X and Y independent ?

4. The joint distribution of continuous random variables X and Y is specified by joint pdf $f_{X,Y}$, with joint range $\mathbb{X}^{(2)} = \{(x, y) : x > 0,\ 0 < y < \exp\{-\beta x^{\alpha}\}\} \subset \mathbb{R}^2$ for parameters $\alpha, \beta > 0$. Suppose that

$$f_{X,Y}(x, y) = c x^{\alpha - 1}$$

for $(x, y) \in \mathbb{X}^{(2)}$, and zero otherwise, for constant c. Find the marginal pdfs of X and Y.

5. Suppose that X and Y are continuous random variables. Are the two functions F_1 and F_2 defined below valid joint cdfs for X and Y ?

$$F_1(x, y) = \begin{cases} 1 - e^{-x-y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad F_2(x, y) = \begin{cases} 1 - e^{-x} - xe^{-y} & 0 \leq x \leq y \\ 1 - e^{-y} - ye^{-x} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

6. Suppose that X and Y have joint pdf that is *constant* on the range $\mathbb{X}^{(2)} = (0, 1) \times (0, 1)$, and zero otherwise. Find the marginal pdf of random variables

$$\text{(i) } U = X/Y \qquad \text{(ii) } V = -\log(XY).$$

stating clearly the range of the transformed random variable in each case.

7. Suppose that X and Y are discrete random variables with joint mass function given by

$$f_{X,Y}(x,y) = c\,\frac{2^{x+y}}{x!\ y!} \qquad x,\ y = 0,1,2,....$$

and zero otherwise, for some constant c. Find the value of c, and the marginal mass functions of X and Y. Prove that X and Y are independent random variables.

8. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = c(x+y) \qquad 0 \le x \le y \le 1$$

and zero otherwise, for some constant c. Find the value of c, the marginal pdfs of X and Y, the joint cdf of X and Y, and the conditional pdf of X given Y.

[*Hint: for the joint cdf, consider $F_{X,Y}(x,y)$ for each element of the following partition of $\mathbb{R}^2$: (i) $x < 0, y < 0$, (ii) $0 \le x \le y$, (iii) $0 < y < 1, x < y$, (iv) $0 < x < 1, y > 1$, and (v) $x, y > 1$.*]

9. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = \frac{1}{2x^2 y} \qquad 1 \le x < \infty,\ 1/x \le y \le x$$

and zero otherwise. Derive the marginal pdf of X, the marginal pdf of Y, the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$. Calculate the marginal expectation of Y, $E_{f_Y}[Y]$. [Hint: First sketch the region for which the joint pdf is strictly positive]

10. Suppose that X, Y and Z are continuous random variables with joint pdf given by

$$f_{X,Y,Z}(x,y,z) = c \qquad 0 < x < y < z < 1$$

and zero otherwise, for some constant c. Find the value of c, and

(a) the joint pdf of X and Z
(b) the joint pdf of Y and Z
(c) the conditional pdf of Y given $X = x$ and $Z = z$.
(d) the conditional pdf of X given $Y = y$ and $Z = z$.
(e) the joint conditional pdf of X and Y given $Z = z$.

11. Suppose that continuous random variables X_1 and X_2 have joint pdf that is constant on the unit square, and zero otherwise. Find the pdf and cdf of $Z = X_1 - X_2$.

12. Suppose that continuous random variables X_1, X_2, X_3 are independent, and have marginal pdfs specified by

$$f_{X_i}(x_i) = c_i x_i^i\, \mathrm{e}^{-x_i} \qquad x_i > 0$$

for $i = 1, 2, 3$.

Find the joint pdf of random variables Y_1, Y_2, Y_3 defined by

$$Y_1 = \frac{X_1}{\sum_{i=1}^{3} X_i} \qquad Y_2 = \frac{X_2}{\sum_{i=1}^{3} X_i} \qquad Y_3 = \sum_{i=1}^{3} X_i$$

and evaluate the marginal expectation of Y_1.