M2S1 - EXERCISES 1

1. Consider events E and F in sample space Ω. Show that

   (i) $E \cap \emptyset = \emptyset$, $E \cup \emptyset = E$

   (ii) $E \cap \Omega = E$, $E \cup \Omega = \Omega$

   (iii) $(E \cap F)^{'} = E^{'} \cup F^{'}$, $(E \cup F)^{'} = E^{'} \cap F^{'}$,

   (iv) $E \subseteq F \Longrightarrow E^{'} \supseteq F^{'}$

   (v) $E \subseteq F \Longrightarrow E \cap F = E$, $E \cup F = F$

Hint: For two events A, B, to show $A = B$ you must show that $\omega \in A \Longleftrightarrow \omega \in B$.

2. Suppose that sample space Ω comprises a finite list of elements, that is, $\Omega = \{\omega_1, ..., \omega_k\}$, say. How many distinct events can be defined on Ω ?

3. Consider sample space Ω. What is the smallest algebra of subsets of Ω ? (that is, what is the smallest set of subsets of Ω for which the three conditions for a set to be an algebra hold ?)

Hint: consider the three conditions (I), (II), (III) in turn.

4. Consider sample space Ω, and an algebra of subsets of Ω, $\mathcal{A} = \{A_1, A_2, ...\}$. Show that if $\mathcal{A}$ is countably additive (closed under *countable* union), then $\mathcal{A}$ is finitely additive (closed under *finite* union).

5. (i) A coin is tossed once. Write down the sample space Ω, and find an algebra of subsets of Ω.

   (ii) Suppose that sample space Ω is countable. Show that the set of all subsets of Ω is a σ-algebra.

6. (a) Consider a measurement experiment where sample outcomes can take any positive real value, so that $\Omega = \mathbb{R}^+$. Consider the events $\{A_k\}$ defined, for integer $k \geq 1$, by saying that A_k occurs if the measurement lies in the interval $(k-1, k]$".

Verify that $\{A_k\}$ forms a partition of Ω, and that the set $\mathcal{A}$ whose elements are countable unions of elements of the sequence $\{A_k\}$, plus $\emptyset$, is a σ-algebra.

(b) Consider a measurement experiment where sample outcomes can take any real value, so that $\Omega = \mathbb{R}$. Consider the collection of events (intervals) of the form $A_i = (-\infty, a_i]$ where a_i is a real number.

(i) Show that all real intervals of the form

$$[a, b], (a, b], [a, b), (a, b) \qquad a, b \in \mathbb{R}$$

can be expressed as unions and intersections of intervals of the form of A_i, and their complements

(ii) Show that the set $\mathcal{A}$ whose elements are countable unions or countable intersections of intervals of the form of A_i and their complements, plus $\emptyset$, is a σ-algebra.