M2S1 : ASSESSED COURSEWORK 3 : SOLUTIONS

1.(a) Following the multivariate transformation theorem, we have $U = XY$ and $V = X$, and hence in the notation of the handouts

$$g_1(t_1, t_2) = t_1 t_2 \qquad g_2(t_1, t_2) = t_1$$

with inverse transformations $X = V$, $Y = U/V$ so that

$$g_1^{-1}(t_1, t_2) = t_2 \qquad g_2(t_1, t_2) = t_1/t_2$$

and hence

$$\frac{\partial}{\partial t_1}\left\{{g_1}^{-1}(t_1, t_2)\right\} = 0 \qquad \frac{\partial}{\partial t_2}\left\{{g_1}^{-1}(t_1, t_2)\right\} = 1$$

$$\frac{\partial}{\partial t_1}\left\{{g_2}^{-1}(t_1, t_2)\right\} = 1/t_2 \quad \frac{\partial}{\partial t_2}\left\{{g_2}^{-1}(t_1, t_2)\right\} = -t_1/t_2^2$$

giving Jacobian $J(u, v) = |0.(-u/v^2) - 1.(1/v)| = |v|^{-1}$ and hence by the theorem

$$f_{U,V}(u, v) = f_{X,Y}(v, u/v)|v|^{-1} \qquad 0 < u < v < 1$$

and hence

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\ dv = \int_{-\infty}^{\infty} f_{X,Y}(v, u/v)|v|^{-1}\ dv \equiv \int_{-\infty}^{\infty} f_{X,Y}(x, u/x)|x|^{-1}\ dx$$

[4 MARKS]

Using this result for the given joint density, where random variables X, Y take values on the unit square, we have

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{X,Y}(v, u/v)|v|^{-1}\ dv = \int_u^1 f_{X,Y}(v, u/v)|v|^{-1}\ dv \\ &= \int_u^1 c(\alpha, \beta)\ v^{\alpha-1}(1-v)^{\beta-1}\ \ c(\alpha+\beta, \gamma)\ \left(\frac{u}{v}\right)^{\alpha+\beta-1}\ \left(1-\frac{u}{v}\right)^{\gamma-1}\ v^{-1}\ dv \end{aligned}$$

as we have the density being positive only on the interval $u < v < 1$ for fixed $0 < u < 1$, as $XY < X$. To compute this integral

$$\begin{aligned} f_U(u) &= \int_u^1 c(\alpha, \beta)\ v^{\alpha-1}(1-v)^{\beta-1}\ \ c(\alpha+\beta, \gamma)\ \left(\frac{u}{v}\right)^{\alpha+\beta-1}\ \left(1-\frac{u}{v}\right)^{\gamma-1} \frac{1}{v}\ dv \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)\int_u^1 \left(\frac{u}{t}\right)^{\alpha-1} \left(1-\frac{u}{v}\right)^{\beta-1} t^{\alpha+\beta-1}(1-t)^{\gamma-1}\ \frac{t}{u}\ \frac{u}{t^2}\ dt \qquad (t = u/v) \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)\int_0^{1-u} \left(\frac{u}{s+u}\right)^{\alpha-1} \left(1-\frac{u}{s+u}\right)^{\beta-1} (s+u)^{\alpha+\beta-2}(1-s-u)^{\gamma-1}\ ds \qquad (s = t-u) \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)\int_0^{1-u} u^{\alpha-1}s^{\beta-1}(1-s-u)^{\gamma-1}\ ds \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)\int_0^{1} u^{\alpha-1}r^{\beta-1}(1-u)^{\beta-1}(1-u)^{\gamma-1}(1-r)^{\gamma-1}(1-u)\ dr \qquad (r = s/(1-u)) \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 r^{\beta-1}(1-r)^{\gamma-1}\ dr \\ &= c(\alpha, \beta)c(\alpha+\beta, \gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{1}{c(\beta, \gamma)} = c(\alpha, \beta+\gamma)u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0 < u < 1 \end{aligned}$$

[6 MARKS]

(b) To compute the covariance need first the marginal expectations of X and Y. Now, for $0 < x < 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} cxy(1-x-y)\ dy = cx\int_0^{1-x} y(1-x-y)\ dy \\ &= cx(1-x)^3 \int_0^1 t(1-t)\ dt \qquad (t = y/(1-x)) \\ &= \frac{c}{6}x(1-x)^3 \qquad 0<x<1 \end{aligned}$$

and

$$\int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^1 \frac{c}{6}x(1-x)^3\ dx = 1 \Longrightarrow c = 120$$

and hence

$$\begin{aligned} f_X(x) &= 20x(1-x)^3 \qquad 0<x<1 \qquad \Longrightarrow \qquad \mathrm{E}_{f_X}[\ X\ ] = \int_0^1 20x^2(1-x)^2\ dx = \frac{1}{3} \\ f_Y(y) &= 20y(1-y)^3 \qquad 0<y<1 \qquad \Longrightarrow \qquad \mathrm{E}_{f_Y}[\ Y\ ] = \frac{1}{3} \end{aligned}$$

by symmetry. Also

$$\begin{aligned} \mathrm{E}_{f_{X,Y}}[\ XY\ ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)\ dxdy = \int_0^1 \left\{\int_0^{1-y} 120x^2y^2(1-x-y)\ dx\right\} dy \\ &= \int_0^1 120y^2 \left\{\int_0^{1-y} x^2(1-x-y)\ dx\right\} dy \\ &= \int_0^1 120y^2 \left[\frac{x^3}{3}(1-y) - \frac{x^4}{4}\right]_0^{1-y} dy \\ &= \int_0^1 10y^2(1-y)^4\ dy \\ &= 10\left[\frac{y^3}{3} - y^4 + \frac{6y^5}{5} - \frac{4y^6}{6} + \frac{y^7}{7}\right]_0^1 = 10\left(\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right) = \frac{2}{21} \end{aligned}$$

and hence

$$\mathrm{Cov}_{f_{X,Y}}[\ X,Y\ ] = \mathrm{E}_{f_{X,Y}}[\ XY\ ] - \mathrm{E}_{f_X}[\ X\ ]\ .\ \mathrm{E}_{f_Y}[\ Y\ ] = \frac{2}{21} - \frac{1}{3}\cdot\frac{1}{3} = -\frac{1}{63}$$

[10 MARKS]

2.(a)(i) To calculate the mgf

$$M_X(t) = \mathrm{E}_{f_X}[\, e^{tX} \,] = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-t)^2}{2}\right\}\, dx$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}\, dy = e^{t^2/2}$$

completing the square in x, and then setting $y = x - t$, as the integrand is a pdf.

[3 MARKS]

(ii) Using the transformation theorem for univariate, 1-1 transformations we have $Y = \sigma X + \mu \iff X = (Y - \mu)/\sigma$, so

$$f_Y(y) = f_X((y-\mu)/\sigma) \,.\, \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \qquad y \in \mathbb{R}$$

[2 MARKS]

To calculate the mgf of Y, use the mgf result given in lectures

$$M_Y(t) = e^{\mu t} M_X(\sigma t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$$

[2 MARKS]

(iii) If $S = X_1 + \ldots X_k$, then by the mgf result given in lectures

$$M_S(t) = \prod_{i=1}^{k} M_{X_i}(t) = \{M_X(t)\}^k = e^{kt^2/2}$$

which is identical to the mgf found in (ii) if $\mu = 0$, $\sigma^2 = k$. Hence by the uniqueness properties of mgfs,

$$f_S(s) = \frac{1}{\sqrt{2\pi k}} \exp\left\{-\frac{s^2}{2k}\right\} \qquad s \in \mathbb{R}$$

[3 MARKS]

(b)(i) If $T = X^2$, then from first principles

$$F_T(t) = \mathrm{P}[\, T \leq t \,] = \mathrm{P}[\, X^2 \leq t \,] = \mathrm{P}[\, -\sqrt{t} \leq X \leq \sqrt{t} \,]$$

$$\Longrightarrow f_T(t) = \frac{1}{2\sqrt{t}} \left[f_X(\sqrt{t}) + f_X(-\sqrt{t})\right] = \frac{1}{\sqrt{2\pi}} t^{-1/2} \exp\left\{-\frac{t}{2}\right\} \qquad t > 0$$

[3 MARKS]

(ii)

$$M_T(t) = \mathrm{E}_{f_T}[\, e^{tT} \,] = \int_{-\infty}^{\infty} e^{tx} f_T(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{x}{2}\right\}\, dx = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{(1-2t)x}{2}\right\}\, dx$$

$$= \left(\frac{1}{1-2t}\right)^{1/2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\}\, dy = \left(\frac{1}{1-2t}\right)^{1/2}$$

where $y = (1-2t)x$, as the integrand is a pdf.

[4 MARKS]

(iii) Using the result for the sums of i.i.d. random variables; if $T_i = X_i^2$,

$$M_U(t) = \prod_{i=1}^{k} M_{T_i}(t) = \{M_T(t)\}^k = \left(\frac{1}{1-2t}\right)^{k/2}$$

[3 MARKS]

SUPPLEMENTARY QUESTIONS

S1. If $Y_2 = \max\{X_1, X_2\}$ then

$$\begin{aligned} \mathrm{P}[\ Y_2 \leq c\ ] &= \mathrm{P}[\ \max\{X_1, X_2\} \leq c\ ] = \mathrm{P}[\ (X_1 \leq c) \cap (X_2 \leq c)\ ] = \mathrm{P}[\ X_1 \leq c\ ]\mathrm{P}[\ X_2 \leq c\ ] = p^2 \\ \Longrightarrow \mathrm{P}[\ Y_2 > c\ ] &= 1 - p^2 \end{aligned}$$

S2. If $Y_1 = \min\{X_1, ..., X_k\}$, then

$$\begin{aligned} F_{Y_1}(y_1) &= \mathrm{P}[\ Y_1 \leq y_1\ ] = \mathrm{P}[\ \min\{X_1, ..., X_k\} \leq y_1\ ] \\ &= 1 - \mathrm{P}[\ \min\{X_1, ..., X_k\} > y_1\ ] = 1 - \mathrm{P}[\ (X_1 > y_1) \cap ... \cap (X_k > y_1)\ ] \\ &= 1 - \mathrm{P}[\ (X_1 > y_1)\ ]...\mathrm{P}[\ (X_k > y_1)\ ] = 1 - (1 - F_X(y_1))...(1 - F_X(y_1)) = 1 - \{(1 - F_X(y_1))\}^k \end{aligned}$$

Now here, $f_X(x) = \lambda e^{-\lambda x}$, so $F_X(x) = 1 - e^{-\lambda x}$, when $x > 0$, and hence

$$F_{Y_1}(y_1) = 1 - \{(1 - F_X(y_1))\}^k = 1 - \left\{1 - (1 - e^{-\lambda y_1})\right\}^k = 1 - e^{-\lambda y_1} \quad \Longrightarrow \quad f_{Y_1}(y_1) = k\lambda e^{-k\lambda y_1} \qquad y_1 > 0.$$

S3. The cdf for X is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_1^x \frac{1}{t^2}dt = 1 - \frac{1}{x} = \frac{x-1}{x} \qquad 1 \leq x \leq \infty$$

Using Result 1 of the Order Statistics section: the joint distribution of the order statistics $Y_1, ..., Y_k$ derived from $X_1, ..., X_k$ is

$$f_{Y_1,...,Y_k}(y_1, ..., y_k) = k! f_{X_1,...,X_k}(x_1, ..., x_k) = k! \prod_{i=1}^{k} \frac{1}{y_i^2}$$

and by Result 2, or by direct calculation as in S2 above, the marginal pdfs of minimum Y_1 and maximum Y_k are given by

$$\begin{aligned} f_{Y_1}(y_1) &= \frac{k!}{(k-1)!\ 0!}\{F_X(y_1)\}^0 \{1 - F_X(y_1)\}^{k-1} f_X(y_1) &= k\left\{\frac{1}{y_1}\right\}^{k-1} \frac{1}{y_1^2} = \frac{k}{y_1^{k+1}} \\ f_{Y_k}(y_k) &= \frac{k!}{0!\ (k-1)!}\{F_X(y_k)\}^{k-1} \{1 - F_X(y_k)\}^0 f_X(y_k) &= k\left\{\frac{y_k - 1}{y_k}\right\}^{k-1} \frac{1}{y_k^2} = \frac{k(y_k-1)^{k-1}}{y_k^{k+1}} \end{aligned}$$

with support $[1, \infty)$ in each case. If $k = 2n+1$, then

$$\begin{aligned} f_{Y_n}(y_n) &= \frac{(2n+1)!}{(n-1)!\ (n+1)!}\{F_X(y_1)\}^{n-1} \{1 - F_X(y_1)\}^{n+1} f_X(y_n) \\ &= \frac{(2n+1)!}{(n-1)!\ (n+1)!}\left\{\frac{y_n - 1}{y_n}\right\}^{n-1} \left\{\frac{1}{y_n}\right\}^{n+1} \frac{1}{y_n^2} \\ &= \frac{(2n+1)!}{(n-1)!\ (n+1)!}\frac{(y_n-1)^{n-1}}{y_n^{2(n+1)}} \end{aligned}$$

S4. We have $F_X(x) = x$, and $f_X(x) = 1$ for $0 < x < 1$. Either by direct calculation, or by using the Order Statistics Result 2, we have

$$\begin{aligned} F_{Y_k}(y_k) &= \{F_X(y_k)\}^k = y_k^k \\ f_{Y_k}(y_k) &= \frac{k!}{0!\ (k-1)!}\{F_X(y_k)\}^{k-1} \{1 - F_X(y_k)\}^0 f_X(y_k) = k y_k^{k-1} \end{aligned}$$

$$P[\ Y_k \geq 0.99\ ] = 1 - P[\ Y_k < 0.99\ ] = 1 - F_{Y_k}(0.99) = 1 - (0.99)^k$$

and the smallest k such that $1 - (0.99)^k \geq 0.95$ is $k = 295$.

The range, R is defined by $R = Y_k - Y_1$, and by the Order Statistics Result 3,

$$f_{Y_1,Y_k}(y_1, y_k) = \frac{k!}{(k-2)!} f_X(y_1) \{F_X(y_k) - F_X(y_1)\}^{k-2} f_X(y_k) = k(k-1)\{y_k - y_1\}^{k-2} \qquad y_1 < y_k$$

Making the transformation $(Y_1, Y_2) \longrightarrow (R, Z)$ where $Z = Y_1$, and applying the multivariate transformation result

$$f_{R,Z}(r, z) = f_{Y_1,Y_k}(z, z + r)\ J(r, z) = k(k-1)\{z + r - z\}^{k-2} = k(k-1)r^{k-2} \qquad 0 < r, z < 1,\ 0 < r + z < 1.$$

as the Jacobian of the transformation, $J(r, z)$, is 1. Finally,

$$f_R(r) = \int_{-\infty}^{\infty} f_{R,Z}(r, z)\ dz = \int_0^{1-r} k(k-1)r^{k-2}\ dz = k(k-1)r^{k-2}(1-r) \qquad 0 < r < 1$$

S5. For this pdf, we have cdf $F_X(x) = x^5,\ 0 < x < 1$. From S2, we have that

$$F_{Y_1}(y_1) = 1 - \{1 - F_X(y_1)\}^k = 1 - (1 - y_1^5)^k \qquad 0 < y_1 < 1.$$

Therfore, $P[\ Y_1 \leq 0.75\ ] = 1 - (1 - (0.75)^5)^k$, which takes the values 0.556, 0.662, 0.742 for $k = 3, 4, 5$. Also

$$f_{Y_1}(y_1) = k\{1 - F_X(y_1)\}^{k-1} f_X(y_1) = 5ky_1^4\{1 - y_1^5\}^{k-1} \qquad 0 \leq y_1 \leq 1$$

and

$$F_{Y_1}(y_1) = 1 - \{1 - y_1^5\}^k = 1 - (1 - y_1^5)^k \quad \longrightarrow \quad \begin{cases} 0 & y_1 = 0 \\ 1 & y_1 > 1 \end{cases}$$

as $k \longrightarrow \infty$, so that the limiting distribution is a step function, with the step at zero. That is, the limiting distribution has mass one at $Y_1 = 0$.

S6. By definition of mgfs for discrete variables, we can deduce immediately that, as

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x)$$

$P[\ X = x\ ]$ is just the coefficient of e^{tx} in the expression for M_X, and hence $P[\ X = 1\ ] = 1/8$, $P[\ X = 2\ ] = 1/4$ and $P[\ X = 3\ ] = 5/8$. Also, we have $E_{f_X}[\ X^r\ ] = M_X^{(r)}(0)$, so that

$$E_{f_X}[\ X\ ] \ = M_X^{(1)}(0) = \frac{1}{8} + 2\frac{1}{4} + 3\frac{5}{8} = \frac{5}{2} \qquad E_{f_X}[\ X^2\ ] = M_X^{(2)}(0) = \frac{1}{8} + 4\frac{1}{4} + 9\frac{5}{8} = \frac{27}{4}$$

$$\Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] \ = E_{f_X}[\ X^2\ ] - \{E_{f_X}[\ X\ ]\}^2 = \frac{1}{2}$$

S7. If $X \sim Bin(n, \theta)$,

$$M_X(t) = (1 - \theta + \theta e^t)^n = (1 + (e^t - 1)\theta)^n = \left(1 + \theta\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right)\right)^n$$

and from the mgf definition $E_{f_X}[\ X^r\ ]$ is $r!$ times the coefficient of t^r in this expansion. Difficult to identify this general term, but can easily identify the coefficient of t as $n\theta = E_{f_X}[\ X\ ]$, and the coefficient of t^2 as $n\theta + n(n-1)\theta^2 = E_{f_X}[\ X^2\ ]$ etc.

S8. For this pdf,

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\ dx = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\ dx = \int_{-2}^{\infty} e^{-(1-t)(x-2)}\ dx \\ &= \frac{e^{-2}}{1-t}\int_{-2(1-t)}^{\infty} e^{-y}\ dy \ = \ \frac{e^{-2}}{1-t}\ \left[-e^{-y}\right]_{-2(1-t)}^{\infty} \ = \ \frac{e^{-2t}}{1-t} \qquad t<1 \end{aligned}$$

Now

$$\begin{aligned} M_X^{(1)}(t) &= \frac{e^{-2t}}{(1-t)^2}(2t-1) \Longrightarrow M_X^{(1)}(0) = -1 = \mathrm{E}_{f_X}[\ X\ ] \\ M_X^{(2)}(t) &= \frac{e^{-2t}}{(1-t)^3}\left[1+(2t-1)^2\right] \Longrightarrow M_X^{(2)}(0) = 2 = \mathrm{E}_{f_X}[\ X^2\ ] \Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] = 1 \end{aligned}$$

S9. We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathrm{E}_{f_X}[\ X\ ]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = \mathrm{E}_{f_X}[\ X^2\ ] - \left\{\mathrm{E}_{f_X}[\ X\ ]\right\}^2 = \mathrm{Var}_{f_X}[\ X\ ]$$