M2S1 - ASSESSED COURSEWORK 4

To be handed in no later than Thursday, 17th December

1. (a)(i) Suppose that X and Y are independent and identically distributed $N(0,1)$ random variables, so that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(x^2+y^2\right)\right\} \qquad x,y \in \mathbb{R}$$

Let random variable U be defined by $U = X/Y$. Show that

$$f_U(u) = \frac{1}{\pi}\frac{1}{1+u^2} \qquad u \in \mathbb{R}$$

[Hint: consider the joint transformation $(X,Y) \longrightarrow (U,V)$ *where* $V = Y$.*]*

(ii) Suppose now that S is a random variable, independent of X and Y, with pdf given by

$$f_s(s) = \left(\frac{1}{2}\right)^{\nu/2}\frac{1}{\Gamma\left(\frac{\nu}{2}\right)}s^{\nu/2-1}e^{-s/2} \qquad s > 0$$

where ν is a positive integer, so that $S \sim Gamma(\nu/2, 1/2) \equiv \chi^2_\nu$. Find the pdf of random variable T defined by $T = X/\sqrt{S/\nu}$.

[Hint: consider the joint transformation $(X,S) \longrightarrow (T,R)$ *where* $R = S$.*]*

(b) Suppose now that the joint pdf of random variables X and Y is specified via the conditional density $f_{X|Y}$ and the marginal density f_Y as

$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}}\exp\left\{-\frac{yx^2}{2}\right\} \qquad x \in \mathbb{R} \qquad f_Y(y) = \left(\frac{\nu}{2}\right)^{\nu/2}\frac{1}{\Gamma\left(\frac{\nu}{2}\right)}y^{\nu/2-1}e^{-\nu y/2} \qquad y > 0$$

where ν is a positive integer, so that $X|Y = y \sim N(0, y^{-1})$ and $Y \sim Gamma(\nu/2, \nu/2)$. Find the marginal pdf of X.

2.(a) Suppose that $X \sim Poisson(\lambda)$, and let Z_1 be defined by

$$Z_1 = \frac{X-\lambda}{\sqrt{\lambda}}$$

Show that the mgf of Z_1 is given by $M_{Z_1}(t) = \exp\left\{-\sqrt{\lambda}t + \lambda\left(e^{t/\sqrt{\lambda}} - 1\right)\right\}$ and hence that, as $\lambda \longrightarrow \infty$, $M_{Z_1}(t) \longrightarrow \exp\left\{t^2/2\right\}$, so that $Z_1 \longrightarrow Z \sim N(0,1)$.

(b) Now suppose that $X \sim Gamma(\lambda, 1)$. For Z_2 defined by

$$Z_2 = \frac{X-\lambda}{\sqrt{\lambda}}$$

Find the mgf of Z_2, and hence identify the distribution of Z_2 as $\lambda \longrightarrow \infty$

You may quote without proof any mgf results given in lecture notes. The following power series expansions may be useful:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \qquad \log(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \ldots$$

SUPPLEMENTARY QUESTIONS

(not to be handed in)

S1. Suppose that X and Y are independent random variables with that $X \sim Gamma(m, \beta)$, and $Y \sim Beta(n, m-n)$, where m and n are positive integers satisfying $n \leq m$. Find the mgf of U defined by $U = XY$, and hence identify the distribution of U, by using the law of iterated expectation;

$$M_U(t) = \mathrm{E}_{f_U}\left[\ e^{tU}\ \right] = \mathrm{E}_{f_{X,Y}}\left[\ e^{tXY}\ \right] = \mathrm{E}_{f_Y}\left[\ \mathrm{E}_{f_{X|Y}}\left[\ e^{tyX} | Y = y\ \right]\ \right] = \mathrm{E}_{f_Y}\left[\ \mathrm{E}_{f_X}\left[\ e^{tyX}\ \right]\ \right]$$

Hint: write out the final integral with respect to y in full, and then make the substitution $v = 1/y$ in this integral. Then integrate by parts to obtain a recursion formula in terms of m and n.

S2. Suppose that X and Y are independent $N(0,1)$ random variables. Using mgfs, and the iterated expectation technique described in S1, find the mgf of $U = XY$.

S3. Suppose that $X_1, ..., X_n$ are a random sample from pdf given by

$$f_X(x) = \frac{(\alpha+1)x^\alpha}{\theta^{\alpha+1}} \quad 0 \leq x \leq \theta$$

and zero otherwise, for parameters $\alpha, \theta > 0$. Let $Y_1, ..., Y_n$ be the corresponding order statistics derived from $X_1, ..., X_n$. By considering cdfs, find the distributions of Y_1 and Y_n as $n \longrightarrow \infty$.

S4. Suppose now that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Define statistics T_1 and T_2 by

$$T_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad T_2 = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Show that $\mathrm{E}_{f_{T_1}}[T_1] = \mathrm{E}_{f_{T_2}}[T_2] = \lambda$.

S5. Suppose that the number of patients that arrive at a hospital casualty department is a Poisson random variable with parameter λ. Suppose also that a patient is either female with probability θ, or male with probability $1-\theta$. Show that the numbers of females and males that arrive are independent Poisson random variables with parameters $\lambda\theta$ and $\lambda(1-\theta)$ respectively.

S6. Prove that the inter-event times of events in a Poisson process with parameter λ, denoted, $X_1, X_2, ...$, are independent random variables having $Exp(\lambda)$ distribution, and hence find the distribution of random variable $S_n = X_1 + ... + X_n$. Identify the limiting distribution of $Z_n = (\lambda S_n - n)/\sqrt{n}$ as $n \longrightarrow \infty$.

Hint: if $X(t)$ is a Poisson process with parameter λ, then the number of events, $N(t)$, that occur in the time interval $(0,t)$ follows a $Poisson(\lambda t)$ distribution. For $t > 0$, identify $\mathrm{P}\left[\ N(t) = 0\ \right]$.

S7. If $X_n \sim \chi^2_n$ for $n = 1, 2, ...$, find the distrbution of $Z_n = \dfrac{X_n - n}{\sqrt{2n}}$ as $n \longrightarrow \infty$.

S8. Let V_n be defined in terms of from n independent $N(\mu, \sigma^2)$ random variables by $V_n = (n-1)\sum_{i=1}^{n} (X_i - \bar{X})^2/\sigma^2$ or $V_n = (n-1)s_n^2/\sigma^2$. Show that $V_n \sim \chi^2_{n-1}$, and that, as $n \longrightarrow \infty$,

$$\frac{\sqrt{n-1}(s_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \longrightarrow Z \sim N(0,1)$$

and hence deduce that,

$$s_n^2 \longrightarrow N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$